Modularity and Web Ontologies

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Abstract

Modularity in ontologies is key both for large scale ontology development and for distributed ontology reuse on the Web. However, the problems of formally characterizing a modular representation, on the one hand, and of automatically identifying modules within an OWL ontology, on the other, has not been satisfactorily addressed, although their relevance has been widely accepted by the Ontology Engineering and Semantic Web communities.

In this paper, we provide a notion of modularity grounded on the semantics of OWL-DL. We present an algorithm for automatically identifying and extracting modules from OWL-DL ontologies, an implementation and some promising empirical results on real-world ontologies.

1 Introduction and Motivation

In Ontology Engineering, as in Software Engineering, modularity is a much praised virtue. Modular representations (or programs) are easier to understand, verify, debug, extend, reuse parts of, and thus facilitate collaborative development. For Web ontologies, where the collaboration is, in large part, uncoordinated, it is often not enough that the ontology be modular in a general sense, but that, for a large ontology, there are extractable parts that can be reused outside the context of the original ontology. Furthermore, there is the expectation that those fragments are not *arbitrary*, but maintain some relation to the meaning of those parts in the original context. Indeed, if the fragments are "modules", one would expect that their extraction preserves key aspects of their embedded meaning.

However, the problems of *formally* characterizing a modular representation, on the one hand, and of automatically identifying modules within an ontology, on the other, have not been satisfactorily addressed in the Ontology Engineering and Semantic Web literature, although their relevance has been widely accepted by those communities.[1] Basic to a clear notion of modular decomposition of a logical theory (such as an ontology) is an account of the the *correctness* of that decomposition. In [2], James Garson proposed a criterion of validity for fragments of a logical theory. A fragment is a *logical module* of a theory just if, for some background logic:

- It is *locally correct*, i.e. any sentence provable in the fragment should be provable in the global theory
- It is *locally complete*, i.e. every sentence in the fragment's "domain" that is provable in the global theory should be provable in the fragment.

The intuition is simple: modular fragments of a theory should entail all and only the entailments regarding the "subject matter" of the fragment that the original theory entailed. That is, modules should have disjoint sets of substantive[2] entailments such that the union of those sets is exactly the set of entailments of the original theory. Of course, there are other desiderata for modules:

- Modules should be *small* and thus, one hopes, faster to reason with; at least they should be proper fragments of the parent theory.
- Modules should be *independent* of each other, that is, the addition or deletion of a module should not affect the others.

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[1] Recent approaches to the problem include [8], [6].

[2] This qualifier is important as, for typical consequence relations, every module will entail all tautologies.

- Modules should be *intelligible*, that is, they should make sense to ontology engineers seeking to (re)use them (and, of course, smallness and independence help with this).

Unfortunately, First Order Logic (FOL) theories are not, generally speaking, modular. Given that, in FOL, contradictions entail everything, it is easy to see that every consistent fragment of an inconsistent ontology will fail to be locally complete. Garson uses this fact, plus the difficulty of determining the consistency of a large evolving FOL theory, to argue that FOL is not a proper logic for modular KR.

However, in a Description Logic (DL) setting where there is a decision procedure for consistency checking that is practical for large, "realistic" KBs it is reasonable to demand that an ontology be consistent. Furthermore, it is not clear that demanding that *every* entailment be preserved, even with the freedom to select some other governing consequence relation, is required for sound and effective reuse.

In traditional DL settings, not all entailments are equally valued. Indeed, there are a set of standard inference services which DL-focused systems expose and emphasize [1]. For ontology engineers designing applications around these service, we can focus on preserving a distinguished subset of the entailments of the parent theory. After all, if the application is certain to only use entailments of a certain sort, then preserving those entailments is sufficient.[3]

Finally, unlike Garson, we are concerned with *reusing* parts of theories, not just improving reasoning performance, but also for the sake of intelligibility for humans and effective reuse. For such a purpose, we claim that the primary focus is on individual *terms* in the signature of an ontology such that each term should be assigned a (single) module. Reusing each term then boils down to retrieving its corresponding module within the ontology.

In this paper, we argue that a notion of modularity that meets most of the requirements listed in this section is achievable and we propose an algorithm for quickly identifying and retrieving modules within an OWL-DL ontology [4]. As a first requirement, we enforce consistency of the OWL theories to be modularized. We show that, for consistent ontologies, local correctness and completeness can be guaranteed with respect to a class of entailments of special relevance in DLs. In particular, we show that the modules we

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[3] This is not an uncommon tactic. For example, it is common to observe that the ground entailments of a positive Datalog program are the same under minimal model and first order semantics.

[4] OWL-DL is a syntactic variant of the expressive DL $\mathcal{SHOIN}(\mathcal{D})$ [7]. For ease of presentation, we restrict ourselves in this paper to $\mathcal{SHOIN}$ and hence do not consider datatypes. Our results, however, can be trivially extended.

identify are locally correct and complete for classification, atomic concept satisfiability and instantiation of atomic classes and roles, among others. We show that modules for consistent ontologies can be identified in polynomial time [5] and that the proposed modularization can be used to optimize DL reasoning, even if the initial KB consistency check cannot be avoided. We provide a principled way to assign a module to each term in the signature of the ontology. Finally, we provide an implementation based on Manchester's OWL-API, a UI in the open source ontology editor Swoop [4] and some empirical results on real-world ontologies.

2 Local Correctness and Completeness

In DL settings, not all entailments are equally valued. DL systems have traditionally focused on solving three main reasoning tasks: **1)** *atomic concept satisfiability* determines whether an atomic concept A in the KB is satisfiable, i.e. if there is a model $\mathcal{I}$ of the KB for which $A^{\mathcal{I}} \neq \emptyset$; **2)** *classification* computes the subsumption partial ordering of all the atomic concepts in the KB; **3)** *instantiation and retrieval* determine whether an individual is an instance of an atomic concept and retrieve all the instances of an atomic concept respectively.

For ontology engineers, it is especially important to ensure that a module extracted from an OWL ontology for re-use or maintenance purposes preserves the results of these reasoning tasks. In other words, if we are to reuse an atomic concept A and retrieve a fragment Γ of the original ontology $\mathcal{O}$ we want to make sure that: A is in the signature of Γ, A is satisfiable in Γ iff it is satisfiable in $\mathcal{O}$, A is subsumed by B in Γ iff it is subsumed by B in $\mathcal{O}$, and finally, a is an instance of A in Γ iff it is an instance of A in $\mathcal{O}$.

We argue that such a fragment Γ of the parent ontology $\mathcal{O}$ reasonably captures the meaning of A in $\mathcal{O}$ and hence it makes perfect sense to retrieve Γ instead of $\mathcal{O}$ whenever we need to reuse A. This intuition naturally yields to the following notion of local correctness and completeness.

Definition 1 *Let $\mathcal{O}$ be an OWL-DL ontology with signature $V = (V_C, V_R, V_I)$ where V_C, V_R, V_I are sets of atomic concepts, atomic roles and individuals respectively. A fragment $\Gamma \subseteq O$ is* ***locally correct and complete*** *for an atomic concept $A \in V_C$ in $\mathcal{O}$ if it verifies the following:*

1. $\Gamma \models (A \sqsubseteq B) \Leftrightarrow \mathcal{O} \models (A \sqsubseteq B)$ *for every* $B \in V_C$
2. $\Gamma \models (B \sqsubseteq A) \Leftrightarrow \mathcal{O} \models (B \sqsubseteq A)$ *for every* $B \in V_C$

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[5] Not counting the consistency check itself.

3. $\Gamma \models A(a) \Leftrightarrow \mathcal{O} \models A(a)$ *for every* $a \in V_I$
4. A *is satisfiable (unsatisfiable) in* Γ *iff it is satisfiable (unsat.) in* $\mathcal{O}$

A fragment $\Gamma \subseteq O$ *is* ***locally correct and complete*** *for an atomic role* $R \in V_R$ *in* $\mathcal{O}$ *if:*

1. $\Gamma \models (P \sqsubseteq Q) \Leftrightarrow \mathcal{O} \models (P \sqsubseteq Q)$ *for every* $Q \in V_R$
2. $\Gamma \models (Q \sqsubseteq P) \Leftrightarrow \mathcal{O} \models (Q \sqsubseteq P)$ *for every* $Q \in V_R$
3. $\Gamma \models Domain(P, A) \Leftrightarrow \mathcal{O} \models Domain(P, A)$ *for every* $A \in V_C$
4. $\Gamma \models Range(P, A) \Leftrightarrow \mathcal{O} \models Range(P, A)$ *for every* $A \in V_C$
5. $\Gamma \models Transitive(P) \Leftrightarrow \mathcal{O} \models Transitive(P)$
6. $\Gamma \models Functional(P) \Leftrightarrow \mathcal{O} \models Functional(P)$
7. $\Gamma \models P(a, b) \Leftrightarrow \mathcal{O} \models P(a, b)$ *for every* $a, b \in V_I$

A fragment $\Gamma \subseteq O$ *is* ***locally correct and complete*** *for an individual* $a \in V_I$ *in* $\mathcal{O}$ *if it verifies the following:*

1. $\Gamma \models A(a) \Leftrightarrow \mathcal{O} \models A(a)$ *for every* $A \in V_C$
2. $\Gamma \models P(a, b) \Leftrightarrow \mathcal{O} \models P(a, b)$ *for every* $P \in V_R, b \in V_I$
3. $\Gamma \models P(b, a) \Leftrightarrow \mathcal{O} \models P(b, a)$ *for every* $P \in V_R, b \in V_I$

The definition above presents several differences with respect to the one proposed by Garson: first, it is specific to Description Logics; second, we do not require the preservation of *all* entailments in the module; third, it is relative to a given term in the signature of the ontology, which is key for reuse tasks.

Given our notion of local completeness and correctness, we will show that there is a class of "safe" OWL-DL theories that *can* be modularized. Interestingly, the safety requirements are not very strict and, in practice, "unsafe" ontologies are not commonly found.

3 The Modularization Algorithm

In this section, we present an algorithm for extracting relevant fragments from the input ontology. We will call the obtained fragments *modules* and compare their properties in terms of the requirements for a modular representation, discussed in Section 1.

The main idea of the algorithm is to generate a *partitioning* of the input ontology $\mathcal{O}$, represented as a directed labeled graph (the *partitioning graph*) and then use the graph to find the module for each term in $\mathcal{O}$.

3.1 The Partitioning Algorithm

In general (see [5], for example), $\{\Gamma_i\}_{1\leq i\leq n}$ is a **partitioning** of a logical theory Γ if $\Gamma = \bigcup_i \Gamma_i$. Each individual Γ_i is called a **partition** and contains a distinct subset of the sentences of Γ.

We represent the partitioning by means of a labeled directed graph $G(\mathcal{O}) = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$. Each node $v_i \in \mathbf{V}$ is labeled with a non-empty partition $\mathcal{L}(v_i) \subseteq \mathcal{O}$. The labels of two different nodes are disjoint ($\mathcal{L}(v_i) \cap \mathcal{L}(v_j) = \emptyset$ for $i \neq j$) and the union of the labels of all the nodes in the graph is precisely $G(\mathcal{O})$ ($\bigcup_{v\in\mathbf{V}} \mathcal{L}(v) = \mathcal{O}$)

Each edge $e = (v_i, v_j)$ is labeled with a non-empty set of roles $\mathcal{L}(e)$ occurring in $\mathcal{O}$ and the labels of different edges are also disjoint ($\mathcal{L}(e) \cap \mathcal{L}(e') = \emptyset$ for $e \neq e'$).

Given two partitions, their respective signatures may *intersect*, thus presence in the signature is not sufficient to determine the "home" partition of a term. We introduce a mapping $\mathcal{V}$ in the graph that assigns to each concept C and individual a occurring in $\mathcal{O}$ a *single* node $\mathcal{V}(C) \in \mathbf{V}$ (respectively $\mathcal{V}(a) \in \mathbf{V}$), and to each role R a single (directed) edge ($\mathcal{V}(R) \in \mathbf{E}$).

Since each term is mapped through $\mathcal{V}$ into a single node or edge, the function $\mathcal{V}$ allows to "disambiguate" the shared symbols. This mapping will reveal key for determining which axioms from the original ontology will be grouped together in the same partition as well as for retrieving the module for each term from the partitioning graph.

The algorithm consists of two main steps: first, a *"safety" check*, which is key for ensuring local correctness and completeness; second, the generation of the partitioning graph $G(\mathcal{O})$. We next explain in detail each step of the algorithm.

1) Safety Check: The presence of certain General Concept Inclusion Axioms (GCIs) may impose general semantic constraints on the ontology *as a whole*. Such constraints may prevent the modularization. In order to detect the presence of "dangerous" GCIs, we introduce the notion of safety:

Definition 2 *Let:* $g : C \in \mathcal{O} \rightarrow \{T, F\}$ *be a function mapping every concept* $C \in \mathcal{O}$ *to a boolean value recursively defined as follows:*

- *If* C *is* $\top$ *or a restriction* $\forall P.D$ *or* $\leq nP$*, then* $g(C) = F$
- *If* C *is an atomic concept, a nominal or a restriction of the form* $\exists P.D$ *or* $\geq nP$*, then* $g(C) = T$
- *If* C *is of the form* $D \sqcap E$*, then* $g(C) = F$ *if* $g(D) = g(E) = F$*, and* $g(C) = T$ *otherwise*
- *If* C *is of the form* $D \sqcup E$*, then* $g(C) = T$ *if* $g(D) = g(E) = T$*, and* $g(C) = F$ *otherwise*

- *If C is of the form $\neg D$, then, if $g(D) = T$, then $g(C) = F$ and, if $g(D) = F$, $g(C) = T$.*

The ontology $\mathcal{O}$ is ***safe*** *iff it contains no axiom of the form $C \sqsubseteq D$ s.t. $g(C) = F$ and $g(D) = T$;*

In order to understand the potential effect of "dangerous" GCIs, let us consider the the following simple ontology, which is *not* safe:

$$\mathcal{O} = \{\top \sqsubseteq \{a\}; \{a\} \sqsubseteq \exists R.C\}$$

Using our algorithm on this ontology (see steps 2 and 3), we would obtain the fragments $\Gamma_1 = \{\top \sqsubseteq \{a\}; \{a\} \sqsubseteq \exists R.C\}$, for a, R and $\Gamma_2 = \{C \sqsubseteq \top\}$, for C. However, Γ_2 is not a suitable module for C, since $\mathcal{O}$ entails $C(a)$ and Γ_2 does not, which breaks local correctness and completeness. In such a case, the algorithm determines that the module for every term is the input ontology itself.

Note that the definition of safety we have provided is *structural* and hence the fact that an ontology does not contain any unsafe axiom does not mean, in principle, that such an axiom cannot be *entailed*, which would compromise our approach. The following theorem shows that unsafe axioms cannot be entailed unless they occur explicitly in the ontology.

Theorem 1 *Let $\mathcal{O}$ be an ontology with signature V that is both consistent and safe, then there are no $\mathcal{SHOIN}$ concepts C,D in V s.t. $g(C) = F$, $g(D) = T$ and $\mathcal{O} \models (C \sqsubseteq D)$.*

2) Generation of the partitioning graph: In case of a positive result in the safety check, the algorithm generates in this step a partitioning graph $G(\mathcal{O})$.

The algorithm performs a succession of *partitioning steps*, as shown in Figure 1. Each step involves *a pair* of nodes in the graph: the node v_0, called the *source* node, which initially contains in its label the input ontology and from which terms and axioms are removed, and a the node v, the *target* node, generated from scratch, to which these are added. Note that the source node is always v_0 and the target node is different is each step.

At the beginning of each partitioning step (see Figure 2), the algorithm selects non-deterministically an term X in the signature of the source node v_0 and changes the value of $\mathcal{V}(X)$. In the case of a concept, for example, $\mathcal{V}(X)$ is updated to v_i, which intuitively means that the concept is "moved" to the target partition.

This initial change will trigger new ones, according to Figure 4.

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-Algorithm Partition($\mathcal{O}$)
-Input: A $\mathcal{SHOIN}$ ontology $\mathcal{O}$
-Output: A partitioning graph $G(\mathcal{O}) = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$

$G(\mathcal{O}) \leftarrow (\{v_0\}, \emptyset, \mathcal{L}, \mathcal{V})$, with:
  $\mathcal{L}(v_0) = \mathcal{O}$
  $\mathcal{V}(X) = v_0$ for each concept or individual $X$ in $\mathcal{O}$
  $\mathcal{V}(P) = (v_0, v_0)$ for each role $P$ in $\mathcal{O}$
if $\mathcal{O}$ not safe, return $G(\mathcal{O})$
for each role $P$ occurring in $\mathcal{O}$, $BoundTo(P) \leftarrow \emptyset$
Repeat
  $G(\mathcal{O}) \leftarrow DoPartitioningStep(G(\mathcal{O}))$
until $\mathcal{L}(v_0) = \emptyset$
$\mathbf{V} \leftarrow \mathbf{V} - v_0$
return $G(\mathcal{O})$
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Figure 1: Partitioning Algorithm

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-Algorithm DoPartitioningStep($G(\mathcal{O})$)
-Input: A partitioning graph $G(\mathcal{O})$
-Output: Updated graph $G(\mathcal{O})$

Create new node $v$ with $\mathcal{L}(v) = \emptyset$ and do $\mathbf{V} \leftarrow \mathbf{V} \cup v$
Select non-deterministically a concept or individual $X$ in $\mathcal{O}$
  with $\mathcal{V}(X) = v_0$, or a role $X$ with $\mathcal{V}(X) = (v_0, v_0)$
if $X$ a concept or individual, then $\mathcal{V}(X) \leftarrow v$
if $X$ a role then $\mathcal{V}(X) \leftarrow (v, v_0)$
$G(\mathcal{O}) \leftarrow MoveTerms(G(\mathcal{O}), v)$
$G(\mathcal{O}) \leftarrow MoveAxioms(G(\mathcal{O}), v)$
return $(G(\mathcal{O}))$
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Figure 2: Partitioning Steps

$\mathcal{O} = \{$	$\mathbf{V} = \{v_1, v_2, v_3, v_4\}$
$St \sqsubseteq \exists enrolledIn.Co$	$\mathbf{E} = \{(v_4, v_1), (v_4, v_3), (v_3, v_4)\}$
$St \sqsubseteq Person$	$\mathcal{L}(v_1) = \{ Co \sqsubseteq \top \}$; $\mathcal{L}(v_2) = \{ Paper \sqsubseteq Pub\}$
$Prof \sqsubseteq \exists teaches.Co \sqcap \exists memberOf.Dept$	$\mathcal{L}(v_3) = \{Dept. \sqsubseteq \exists memberOf^{-}.St\}$;
$Paper \sqsubseteq Pub$	$\mathcal{L}(v_4) = \{ St \sqsubseteq \exists enrolledIn.Co$; $St \sqsubseteq Person$;
$Dept. \sqsubseteq \exists memberOf^{-}.St$	$Prof \sqsubseteq \exists teaches.Co \sqcap \exists memberOf.Dept$
$\exists enrolledIn.\top \sqsubseteq Person\}$	$\exists enrolledIn.\top \sqsubseteq Person\}$

Figure 3: A Decomposition into a Partitioning Graph

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-Algorithm MoveTerms($G(\mathcal{O}), v$)
-Input: A partitioning graph $G(\mathcal{O}) = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$
   The target node $v$ in the current partitioning step
-Output: A partitioning graph with updated mapping $\mathcal{V}$

Repeat
 for all concept $C \neq \top$ with $\mathcal{V}(C) = v_0$
   if any of the following conditions holds:
     1)$(C \sqsubseteq D)$ or $(D \sqsubseteq C) \in \mathcal{L}(v_0)$, and $\mathcal{V}(D) = v$
     2)$C(a) \in \mathcal{L}(v_0)$ and $\mathcal{V}(a) = v$
     3)$\exists P.C$ or $\forall P.C \in \mathcal{L}(v_0)$ and $\mathcal{V}(P) \in \{(v_0, v), (v, v)\}$
     4)$(C \sqcap D)$ or $(C \sqcup D) \in \mathcal{L}(v_0)$ and $\mathcal{V}(D) = v$
     5)$C$ of the form $D \sqcap E$, or $D \sqcup E$ and $\mathcal{V}(E) = v$ or $\mathcal{V}(D) = v$
     6)$C$ a restriction $\exists P.D, \forall P.D, \geq nP$ or $\leq nP$,
       and $\mathcal{V}(P) \in \{(v, v_0), (v, v), (v_j, v_0)\}$
     7)$C$ of the form $\{a\}$ and $\mathcal{V}(a) = v$
     8)$(\neg C) \in \mathcal{L}(v_0)$ and $\mathcal{V}(\neg C) = v$
     9)$C, E \in BoundTo(P)$ and $\mathcal{V}(E) = v$
   then $\mathcal{V}(C) \leftarrow v$
   if 3) has held, then $BoundTo(P) \leftarrow BoundTo(P) \cup \{C\}$
 for all individual $a$ with $\mathcal{V}(a) = v_0$
   if any of the following conditions holds:
     1)$C(a) \in \mathcal{L}(v_0)$ and $\mathcal{V}(C) = v$
     2) $\{a\} \in \mathcal{L}(v_0)$ and $\mathcal{V}(\{a\}) = v$
     3)$P(a, b) \in \mathcal{L}(v_0)$ and $\mathcal{V}(P) \in \{(v, v_0), (v, v), (v_j, v_0)\}$
     4)$P(b, a) \in \mathcal{L}(v_0)$ and $\mathcal{V}(P) \in \{(v_0, v), (v, v)\}$
   then $\mathcal{V}(a) \leftarrow v$
   if 4) has held, then $BoundTo(P) \leftarrow BoundTo(P) \cup \{a\}$
 for all role $P$ with $\mathcal{V}(P) \in \{(v_0, v_0), (v_0, v), (v, v_0), (v_o, v_j)\}$
   if $(P \sqsubseteq Q)$ or $(Q \sqsubseteq P) \in \mathcal{L}(v_0)$,
      and $\mathcal{V}(Q) \in \{(v_0, v), (v, v_0), (v, v), (v_j, v_0)\}$ then $\mathcal{V}(P) \leftarrow \mathcal{V}(Q)$
   if $\mathcal{V}(P) \in \{(v_0, v), (v, v_0)\}$ and $P$ transitive, then $\mathcal{V}(P) \leftarrow (v, v)$
   if $P(a, b) \in \mathcal{L}(v_0)$, $\mathcal{V}(a) = v$,
      and $\mathcal{V}(P) = (v_0, v_0)$, then $\mathcal{V}(P) \leftarrow (v, v_0)$; add $a$ to $BoundTo(P)$
      and $\mathcal{V}(P) = (v_0, v)$, then $\mathcal{V}(P) \leftarrow (v, v)$; remove $a$ from $BoundTo(P)$
      and $\mathcal{V}(P) = (v_0, v_j)$, then $\mathcal{V}(P) \leftarrow (v_j, v_0)$
   if $P(a, b) \in \mathcal{L}(v_0)$ and $\mathcal{V}(b) = v$,
      and $\mathcal{V}(P) = (v_0, v_0)$, then $\mathcal{V}(P) \leftarrow (v_0, v)$; add $b$ to $BoundTo(P)$
      and $\mathcal{V}(P) = (v, v_0)$, then $\mathcal{V}(P) \leftarrow (v, v)$; remove $b$ from $BoundTo(P)$
   if $D$ a restriction on $P$, $\mathcal{V}(D) = (v_0, v)$
      and $\mathcal{V}(P) = (v_0, v_0)$, then $\mathcal{V}(P) \leftarrow (v, v_0)$
      and $\mathcal{V}(P) = (v_0, v)$, then $\mathcal{V}(P) \leftarrow (v, v)$
      and $\mathcal{V}(P) = (v_0, v_j)$, then $\mathcal{V}(P) \leftarrow (v_j, v_0)$
   if $\exists P.C$ or $\forall P.C \in \mathcal{L}(v_0)$, $\mathcal{V}(C) = v$
      and $\mathcal{V}(P) = (v_0, v_0)$, then $\mathcal{V}(P) \leftarrow (v_0, v)$
      and $\mathcal{V}(P) = (v, v_0)$, then $\mathcal{V}(P) \leftarrow (v, v)$
   if $\mathcal{V}(P) = (v_0, v_0)$ and $\mathcal{V}(Inv(P)) = (v_0, v)$, then $\mathcal{V}(P) \leftarrow (v, v_0)$
   if $\mathcal{V}(P) = (v_0, v_0)$ and $\mathcal{V}(Inv(P)) = (v, v_0)$, then $\mathcal{V}(P) \leftarrow (v_0, v)$
   if $\mathcal{V}(P) \in \{(v_0, v_0), (v_0, v), (v, v_0)\}$ and $\mathcal{V}(Inv(P)) = (v, v)$, then $\mathcal{V}(P) \leftarrow (v, v)$
until no change in $\mathcal{V}$ is triggered
return $G(\mathcal{O})$
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Figure 4: Moving Terms

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-Algorithm MoveAxioms($G(\mathcal{O})$,v)
-Input: A partitioning graph $G(\mathcal{O}$
    The target node $v$ in the current partitioning step
-Output: An updated partitioning graph $G(\mathcal{O})$

for each Axiom $\mathcal{A} \in \mathcal{L}(v_0)$
 if $\mathcal{A}$ is of any of the following forms:
   1)$C \sqsubseteq D$ and $\mathcal{V}(C) = \mathcal{V}(D) = v$
   2)$C(a)$ and $\mathcal{V}(a) = \mathcal{V}(C) = v$
   3)$P \sqsubseteq Q$, and $\mathcal{V}(P) = \mathcal{V}(Q)$, with $\mathcal{V}(P) \in \{(v, v_0), (v, v), (v_j, v_0)\}$
   4)$Trans(P)$ and $\mathcal{V}(P) \in \{(v, v_0), (v, v)\}$
   5)$P(a, b)$ and $\mathcal{V}(P) =\in \{(v, v_0), (v, v), (v_j, v_0)\}$
 then $\mathcal{L}(v_0) \leftarrow \mathcal{L}(v_0) - \{\mathcal{A}\}$
   and $\mathcal{L}(v) \leftarrow \mathcal{L}(v) \cup \{\mathcal{A}\}$
for each P, s.t. $\mathcal{V}(P) = (v_0, v)$, do
 $L((v_0, v)) \leftarrow L((v_0, v)) \cup \{P\}$ for each $P \in \mathcal{L}((v_j, v_0))$ with $v_j \neq v$
 if $\forall X \in BoundTo(P)$, $\mathcal{V}(X) = v$, then
   $\mathcal{L}((v_j, v_0)) = \mathcal{L}((v_j, v_0)) - \{P\}$
   if $\mathcal{L}((v_j, v_0)) = \emptyset$ remove edge $(v_j, v_0)$ from $G(\mathcal{O})$
   if $(v_j, v) \notin \mathbf{E}$, add edge $(v_j, v)$ to $G(\mathcal{O})$
   $\mathcal{L}((v_j, v)) = \mathcal{L}((v_j, v)) \cup \{P\}$
return $G(\mathcal{O})$
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Figure 5: Moving Axioms

Depending on the final value of the $\mathcal{V}$ function, some of the axioms in $\mathcal{L}(v_0)$ are removed from $\mathcal{L}(v_0)$ and added to $\mathcal{L}(v)$ and the labels of the edges involving the target and the source nodes are updated.

In Figure 3, we provide the content of the partitioning graph at the end of each partitioning step for an example ontology. The reader should be able to reproduce these results using the the algorithms in Figures 1, 4 and 5.[6]

It is worth noting that, although the initial change in each partitioning step is chosen non-deterministically, it is possible to prove that the result is deterministic and, given an input ontology, the same partitioning graph will always be obtained.

3.1.1 Significance of the Partitioning Graph

It is worth taking a closer look to the partitioning graph generated in Figure 3. The graph contains four partitions $\mathcal{L}(v_1), ..., \mathcal{L}(v_4)$. A quick examination of the axioms they contain reveals that the partitions describe intuitively disjoint subject matters, namely courses, publications, departments and students respectively.

The correspondence of each partition to a well-defined application domain, intuitively disjoint from the rest is a general property of the partitions

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[6] As a remark, the set $BoundTo(P)$ represents the set of terms that are "forced" to end up in the same partition due to the fact that a role P cannot appear in the label of two different edges.

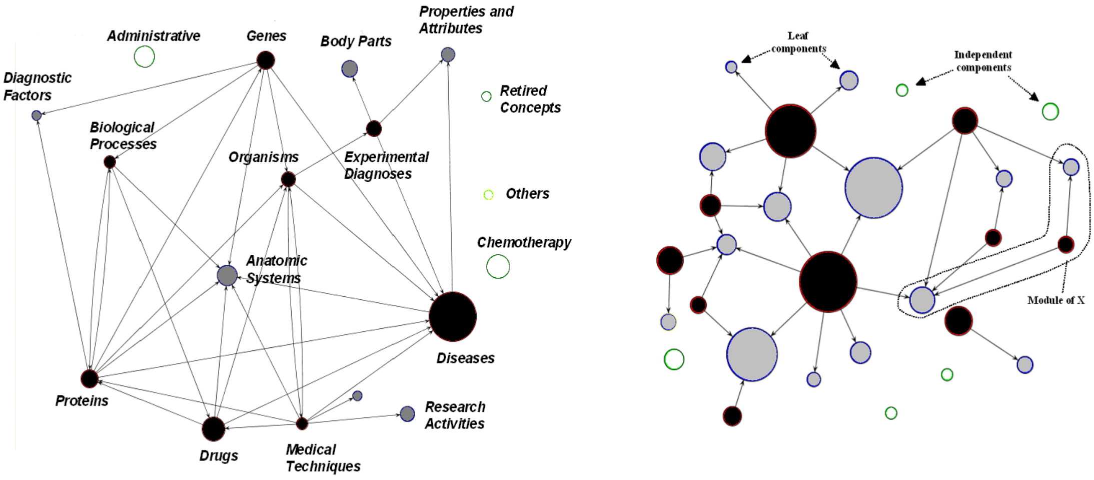

Figure 6: **Partitioning Graph for NCI (left) and OWL-S (right)**

generated using our algorithm and can be observed in large, real-world ontologies.

A prominent example is the NCI [7] Thesaurus [3], a huge, highly structured ontology dealing with the biomedical domain. NCI is a reference terminology covering areas of basic and clinical science, built with the goal of facilitating translational research in cancer.

The decomposition obtained for NCI can be obtained in less than 45 seconds using Swoop and is shown on the left hand side of Figure 6. The figure uses the graph layout in the ontology editor Swoop for visualizing partitioning graphs. In such a layout, the size of the nodes is proportional to the size of the partitions. Isolated nodes are represented in white, leaf nodes in grey and nodes with outgoing edges in black.

The partitions of NCI represent a well-defined sub-domain within the ontology. For example, the knowledge about genes, drugs, medical techniques, etc. are each associated to a different partition. These domains are pairwise disjoint in the sense that they do not share instances (a drug is not a gene and vice-versa). The connections suggest which domains within the ontology are most relevant. For example, highly interconnected partitions, such as the ones dealing with genes and diseases, are central to the ontology. Other nodes, like the one dealing with anatomical structures, are leaves in the graph, and hence represent "secondary" domains.

The following theorem justifies why this fact is generally observed:

Theorem 2 *Let $\mathcal{O}$ be consistent and safe and $G(\mathcal{O}) = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$ with*

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[7] National Cancer Institute

$|\mathbf{V}| = n$, *then there exists a model* $\mathcal{J} = (\mathcal{W}_J, .^{\mathcal{J}})$ *of* $\mathcal{O}$ *such that:*

- $\mathcal{W}_J = \bigcup_{i=1,\ldots,n} \mathcal{W}_i$ *with* $\mathcal{W}_i \cap \mathcal{W}_j = \emptyset$ *for* $i \neq j$*, and* $\mathcal{W}_i \neq \emptyset$
- $A^{\mathcal{J}} \subseteq \mathcal{W}_i$*, for each* $A \in V_C$ *with* $\mathcal{V}(A) = v_i$
- $R^{\mathcal{J}} \subseteq \mathcal{W}_i \times \mathcal{W}_j$*, for each* $R \in V_R$ *with* $\mathcal{V}(R) = (v_i, v_j)$*.*
- $a^{\mathcal{J}} \in \mathcal{W}_i$ *for each* $a \in V_I$ *with* $\mathcal{V}(a) = v_i$

The theorem establishes the existence of a very special family of models for $\mathcal{O}$. These models evaluate each partition in a different *logical* sub-domain, disjoint from the rest. We argue that there exists a very close correspondence between the ability to distinguish disjoint logical sub-domains and the existence of different subject matters within an OWL theory.

The theorem provides an insight about the way the ontology has been modeled. In particular, it suggests one of the following: either the partitions correspond to actual non-overlapping application domains, intended by the ontology engineer, or the ontology is underspecified and some of the partitions correspond to "unused information". In the latter case, these partitions identify parts of the ontology that probably need to be further developed.

Finally, Theorem 2 establishes the basis for designing new optimizations for DL reasoners. The existence of the class of models identified in the theorem makes it possible to identify sentences that *cannot* be entailed by the ontology $\mathcal{O}$:

Theorem 3 *Let* $\mathcal{O}$ *be consistent and safe and* $G(\mathcal{O}) = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$*, the axioms of the following form* ***cannot*** *be entailed by* $\mathcal{O}$*:* ***1)*** $C \sqsubseteq D$*, with* $g(C) = g(D) = T$ *and* $\mathcal{V}(C) \neq \mathcal{V}(D)$*;* ***2)*** $C(a)$ *with* $g(C) = T$ *and* $\mathcal{V}(a) \neq \mathcal{V}(C)$*;* ***3)*** $R \sqsubseteq S$ *with* $\mathcal{V}(R) \neq \mathcal{V}(S)$*;* ***4)*** $R(a, b)$ *with* $\mathcal{V}(R) \neq (\mathcal{V}(a), \mathcal{V}(b))$

As a consequence of the theorem above, many subsumption and realization tests can be avoided.

3.2 Identification and Extraction of Modules

The *module* for each term is obtained from the partitioning graph using the algorithm in Figure 7. According to the Figure, if $\mathcal{V}(X) = v_i$, the **module** for X in $\mathcal{O}$ is the union of all the axioms contained in the nodes that are accessible from v_i through a directed path in $G(\mathcal{O})$. There are cases, however, where the module for X computed that way is not locally correct and complete for X. For example, consider the following ontology:

$$\mathcal{O} = \{C \sqsubseteq \forall R.B \ ; \ B \sqsubseteq E; \ C(a) \ ; \ R(a, b)\}$$

```
-Algorithm GenerateModule($G(\mathcal{O})$, X)
-Input: The partition graph $G(\mathcal{O})$
    An term X in $\mathcal{O}$
-Output: The module $\Gamma$ for X in $\mathcal{O}$

$v \leftarrow \mathcal{V}(X)$
$\Gamma \leftarrow \mathcal{L}(v)$
Add to $\Gamma$ all axioms in the label of the nodes accessible from $v$.
if $\mathcal{L}(v)$ has nominals, then
  for each predecessot $w$ of $v$ in $G(\mathcal{O})$
      Select any term $Z$ in $\mathcal{L}(w)$
      $\Gamma \leftarrow \Gamma \cup GenerateModule(G(\mathcal{O}), Z)$
  return $\Gamma$
for each predecessor $w$ of $v$ s.t. $P(a,b) \in \mathcal{L}(w)$ with $P \in \mathcal{L}((w,v))$
      Select any term $Z$ in $\mathcal{L}(w)$
      $\Gamma \leftarrow \Gamma \cup GenerateModule(G(\mathcal{O}), Z)$
return $\Gamma$
```

Figure 7: Generation of Modules

The partitioning algorithm would generate a graph with two nodes v, w, with $\mathcal{L}(v) = \{C \sqsubseteq \forall R.B; C(a); R(a,b)\}$ and $\mathcal{L}(w) = \{B \sqsubseteq E\}$ connected by an edge (v,w) with $\mathcal{L}((v,w)) = \{R\}$. The module Γ for B would be just $\Gamma = \mathcal{L}(w)$; however $\mathcal{O} \models B(b)$, which is not entailed in Γ, thus violating local correctness and completeness. The problem is caused by the presence of the axiom $R(a,b)$, where R is in the label of the edge connecting v and w. In these cases (see Figure 7) we need to "backtrack" in the graph in order to satisfy local correctness and completeness, and we would add the axioms in $\mathcal{L}(v)$ to the module for B. Similar effects could occur in case w contained nominals. The correctness of our approach is based on the following theorem:

Theorem 4 *The fragment* $\Gamma =$ *GenerateModule(*$G(\mathcal{O})$*, X) is locally correct and complete for an term* X *in* $\mathcal{O}$*.*

It can also be shown that the algorithms described in Figures 1 and 7 are worst-case quadratic in the size of the input ontology and hence the module for an term in a consistent ontology can be obtained in polynomial time

As an example of module extraction from a partitioning graph, consider Figure 6, which shows the decomposition for the OWL-S ontologies, describing Web Services. The ontology exhibits a nice decomposition, since a significant proportion of nodes correspond to independent or leaf nodes (white and grey nodes respectively), which is ideal for re-use. Interestingly, there is a improvement in modularity for every term, in the sense that every module is *strictly smaller* than the ontology as a whole. Finally, note that the whole modularization process is *completely automatic.* No user intervention is required at any stage of the process.

4 Related Work and Conclusion

The problem of modularity in Web ontologies has been recently addressed in [8] and [6]. In [8], the output of the modularization process is presented as a graph visualization of the different kinds of information contained in the input ontology. However, the heuristics used to generate the visualization only consider a small fragment of OWL-DL and no correspondence between the nodes of the graph and sets of axioms is provided. The PromptFactor tool [6] uses structural tracing for extracting relevant fragments of ontologies. Although the output in this case is a set of axioms, a formal characterization of their properties is lacking and hence no notion of correctness of the process is established.

In this paper, we have presented a method for automatically identifying and extracting relevant fragments of ontologies, called modules, with precise semantic guarantees. Our method encompasses the full expressive power of OWL-DL and provides a good computational performance. Our initial experimental results with real-world ontologies show that, for most terms, the modules we obtain can be notably smaller than the original ontology, which facilitates re-use, processability, understandability and maintenance.

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A Appendix: Proofs

A.1 Proof for Theorem 1

Safety is a property of some $\mathcal{SHOIN}$ knowledge bases. In order to provide a formal characterization for it, we start with the notion of *expansion* of the domain of an interpretation.

Definition 3 ***(Domain Expansion)***
Let $\mathcal{I} = (W_{\mathcal{I}}, .^{\mathcal{I}})$ and $\mathcal{J} = (W_{\mathcal{J}}, .^{\mathcal{J}})$ be $\mathcal{SHOIN}$ interpretations such that:

- $W_{\mathcal{J}} = W_{\mathcal{I}} \cup \mathbf{S}$.
- $A^{\mathcal{I}} = A^{\mathcal{J}}$
- $R^{\mathcal{I}} = R^{\mathcal{J}}$
- $a^{\mathcal{I}} = a^{\mathcal{J}}$

where $\mathbf{S}$ is a non-empty set disjoint with $W_{\mathcal{I}}$.
We say that $\mathcal{J}$ has been obtained from $\mathcal{I}$ through a ***domain expansion*** *with set $\mathbf{S}$.*

Note that $\mathcal{I}$ and its expansion $\mathcal{J}$ with set $\mathbf{S}$ *only* differ in the fact that the interpretation domain of $\mathcal{I}$ is a *proper subset* of the domain of $\mathcal{J}$ (with $\mathbf{S}$ being the set of additional domain elements). Aside from this, both interpretations are identical, since their evaluation of atomic concepts, roles and individuals coincide.

Although the additional domain elements provided by $\mathcal{J}$ (i.e., the elements of the set $\mathbf{S}$) do not occur in the interpretation of atomic concepts, roles and individuals, it is not hard to see that they will occur in the interpretation of some complex constructs. For example, if A is an atomic concept, $A^{\mathcal{J}} \cap \mathbf{S} = \emptyset$, but $\mathbf{S} \subseteq (\neg A)^{\mathcal{J}}$.

In order to determine which concepts will contain elements from $\mathbf{S}$ when interpreted by $\mathcal{J}$, we introduce the notion of *invariance under domain expansions.*

Definition 4 ***(Invariance under Domain Expansions)***
A $\mathcal{SHOIN}$-Concept C is ***invariant under domain expansions*** *if, for every possible pair of interpretations $\mathcal{I}, \mathcal{J}$ for C such that $\mathcal{J}$ is obtained from $\mathcal{I}$ through a domain expansion with set $\mathbf{S}$, it holds that $C^{\mathcal{I}} = C^{\mathcal{J}}$.*

We are now ready to provide a rationale to the function g, introduced in the definition of safety.

Lemma 2 *Let C be a $\mathcal{SHOIN}$-Concept, then:*

1. *If $g(C) = T$, then C is invariant under domain expansions.*

2. *If $g(C) = F$ and $\mathcal{J}$ is obtained from $\mathcal{I}$ by a domain expansion with set* $\mathbf{S}$*, then* $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \mathbf{S}$.

Proof

Let $\mathcal{I} = (\mathcal{W}_I, .^{\mathcal{I}})$ be an interpretation for C and $\mathcal{J} = (\mathcal{W}_J, .^{\mathcal{J}})$ an interpretation obtained from $\mathcal{I}$ through a domain expansion with set $\mathbf{S}$.

First, note that, for every $\mathcal{SHOIN}$ abstract role R (an atomic role or its inverse), we have that $R^{\mathcal{J}} = R^{\mathcal{I}}$.

We show that, if $g(C) = T$, then $C^{\mathcal{I}} = C^{\mathcal{J}}$, whereas if $g(C) = F$, $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \mathbf{S}$.

We proceed by induction. At the base of the induction we have the following cases:

- If C is an atomic concept, then $g(C) = T$ and by definition $C^{\mathcal{J}} = C^{\mathcal{I}}$.
- If C is of the form $\{a\}$, then $g(C) = T$ and, since $a^{\mathcal{J}} = a^{\mathcal{I}}$, it follows that $C^{\mathcal{J}} = C^{\mathcal{I}}$.
- If C is the top concept $\top$, then $g(C) = F$ and, since $(\top)^{\mathcal{J}} = \mathcal{W}_J$, $(\top)^{\mathcal{I}} = \mathcal{W}_I$ and $\mathcal{W}_J = \mathcal{W}_I \cup \mathbf{S}$, we have that $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \mathbf{S}$.
- Let C be of the form $\geq nR$. In this case $g(C) = T$. It is easy to see that, since $R^{\mathcal{J}} = R^{\mathcal{I}}$, $R^{\mathcal{I}} \subseteq \mathcal{W}_i \times \mathcal{W}_i$ and $\mathcal{W}_I \cap \mathbf{S} = \emptyset$, also $C^{\mathcal{J}} = C^{\mathcal{I}}$.
- Let C be of the form $\leq nR$. In this case $g(C) = F$. Again, we have that $R^{\mathcal{J}} = R^{\mathcal{I}}$ and $\mathcal{W}_J = \mathcal{W}_I \cup \mathbf{S}$ with $\mathcal{W}_I \cap \mathbf{S} = \emptyset$.

  Let $x \in (\leq nR)^{\mathcal{J}}$ and $x \notin (\leq nR)^{\mathcal{I}} \cup \mathbf{S}$, then $x \in \mathcal{W}_I - (\leq nR)^{\mathcal{I}}$, i.e. $x \in (\geq n+1R)^{\mathcal{I}}$. If $x \in (\geq n+1R)^{\mathcal{I}}$, we have seen above that $x \in (\geq n+1R)^{\mathcal{J}}$ and hence $x \notin (\leq nR)^{\mathcal{J}}$, which yields a contradiction.

  On the other hand, let $x \in (\leq nR)^{\mathcal{I}} \cup \mathbf{S}$ and $x \notin (\leq nR)^{\mathcal{J}}$. If $x \notin (\leq nR)^{\mathcal{J}}$, then $x \in (\geq n+1R)^{\mathcal{J}}$ and hence, as seen above, $x \in (\geq n+1R)^{\mathcal{I}}$. But, if $x \in (\geq n+1R)^{\mathcal{I}}$ then $x \notin (\leq nR)^{\mathcal{I}}$ and $x \notin \mathbf{S}$, because $R^{\mathcal{I}} \subseteq \mathcal{W}_I \times \mathcal{W}_I$.

We now verify the induction step:

- Let C be of the form $\neg D$:
  - If $g(D) = T$, then $g(C) = F$. By induction, $D^{\mathcal{J}} = D^{\mathcal{I}}$. Since $C^{\mathcal{J}} = \mathcal{W}_J - D^{\mathcal{J}}$, $\mathcal{W}_J = \mathcal{W}_I \cup \mathbf{S}$ and $D^{\mathcal{J}} = D^{\mathcal{I}}$, we have that $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \mathbf{S}$.
  - If $g(D) = F$, then $g(C) = T$. By induction, $D^{\mathcal{J}} = D^{\mathcal{I}} \cup \mathbf{S}$. Since $C^{\mathcal{J}} = \mathcal{W}_J - D^{\mathcal{J}}$, $\mathcal{W}_J = \mathcal{W} \cup \mathbf{S}$ and $D^{\mathcal{J}} = D^{\mathcal{I}} \cup \mathbf{S}$, we have that $C^{\mathcal{J}} = C^{\mathcal{I}}$.
- Let C be of the form $D \sqcap E$. By induction, if $g(D) = T$, then $D^{\mathcal{J}} = D^{\mathcal{I}}$, whereas if $g(D) = F$, then $D^{\mathcal{J}} = D^{\mathcal{I}} \cup \mathbf{S}$. Analogously, if $g(E) = T$, $E^{\mathcal{J}} = E^{\mathcal{I}}$ and if $g(E) = F$, then $E^{\mathcal{J}} = E^{\mathcal{I}} \cup \mathbf{S}$. We have the following cases:

  - If $g(D) = F$ and $g(E) = F$, then $g(C) = F$. We have that $C^{\mathcal{J}} = D^{\mathcal{J}} \cap E^{\mathcal{J}} = (D^{\mathcal{I}} \cup \mathbf{S}) \cap (E^{\mathcal{I}} \cup \mathbf{S}) = (D^{\mathcal{I}} \cap E^{\mathcal{I}}) \cup \mathbf{S} = C^{\mathcal{I}} \cup \mathbf{S}$.
  - If either $g(D) = T$ or $g(E) = T$, then $g(C) = T$. It is easy to see that, in any of those cases, $C^{\mathcal{J}} = D^{\mathcal{I}} \cap E^{\mathcal{I}} = C^{\mathcal{I}}$.

- Let C be of the form $D \sqcup E$. The argument is analogous to the above case.

- Let C be of the form $\exists R.D$. In this case, $g(C) = T$. Note that, since $R^{\mathcal{J}} = R^{\mathcal{I}}$, both if $g(D) = T$ (in which case $D^{\mathcal{J}} = D^{\mathcal{I}}$), or if $g(D) = F$ (in which case $D^{\mathcal{J}} = D^{\mathcal{I}} \cup \mathbf{S}$), $C^{\mathcal{J}} = C^{\mathcal{I}}$.

- Let C be of the form $\forall R.D$. In this case $g(C) = F$. We have that $R^{\mathcal{J}} = R^{\mathcal{I}}$ and $\mathcal{W}_J = \mathcal{W}_I \cup \mathbf{S}$ with $\mathcal{W}_I \cap \mathbf{S} = \emptyset$.

  Let $x \in (\forall R.D)^{\mathcal{J}}$ and $x \notin (\forall R.D)^{\mathcal{I}} \cup \mathbf{S}$, then $x \in \mathcal{W}_I - (\forall R.D)^{\mathcal{I}}$, i.e. $x \in (\exists R.\neg D)^{\mathcal{I}}$. Since $R^{\mathcal{I}} = R^{\mathcal{J}}$ and $\mathcal{W}_I \cap \mathbf{S} = \emptyset$, we have that $x \in (\exists R.\neg D)^{\mathcal{J}}$, independently of whether $g(\neg D) = T$, in which case by induction $(\neg D)^{\mathcal{J}} = (\neg D)^{\mathcal{I}}$ or $g(\neg D) = F$, which implies that $(\neg D)^{\mathcal{J}} = (\neg D)^{\mathcal{I}} \cup \mathbf{S}$. Since $x \in (\exists R.\neg D)^{\mathcal{J}}$, then $x \notin \forall R.D)^{\mathcal{J}}$, which yields a contradiction.

  On the other hand, let $x \in (\forall R.D)^{\mathcal{I}} \cup \mathbf{S}$ and $x \notin (\forall R.D)^{\mathcal{J}}$. If $x \notin (\forall R.D)^{\mathcal{J}}$, then $x \in (\exists R.\neg D)^{\mathcal{J}}$. Again, $x \in (\exists R.\neg D)^{\mathcal{I}}$, independently of the value of $g(D)$. But, if $x \in (\exists R.D)^{\mathcal{I}}$, then $x \notin (\forall R.D)^{\mathcal{I}}$, which yields a contradiction.

Q.E.D

From the lemma above, we can obtain a first semantic characterization of safety.

Lemma 3 *Let $\mathcal{O}$ be a consistent $\mathcal{SHOIN}$ ontology and $\mathcal{I} \models \mathcal{O}$. Let $\mathcal{J}$ be obtained from $\mathcal{I}$ through a domain expansion with set $\mathbf{S}$, then $\mathcal{J} \models \mathcal{O}$ iff $\mathcal{O}$ is safe.*

Proof

Let $\mathcal{I} = (\mathcal{W}_I, .^{\mathcal{I}})$ be an interpretation such that $\mathcal{I} \models \mathcal{O}$ and $\mathcal{I} \models (C \sqsubseteq D)$. Let $\mathcal{J} = (\mathcal{W}_J, .^{\mathcal{J}})$ the interpretation obtained from $\mathcal{I}$ by a domain expansion with set $\mathbf{S}$. We show that $\mathcal{J}$ is a model of $\mathcal{O}$ iff $\mathcal{O}$ is safe.

We use Lemma 2, which ensures that, for every concept C in $\mathcal{O}$, if $g(C) = T$, then $C^{\mathcal{I}} = C^{\mathcal{J}}$, whereas $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \mathbf{S}$, if $g(C) = F$.

Let $\mathcal{O}$ be safe, then each of the concept inclusion axioms $C \sqsubseteq D$ in $\mathcal{O}$ are of either of the following forms:

1. $g(C) = T$ and $g(D) = T$. In this case $C^{\mathcal{J}} = C^{\mathcal{I}}$ and $D^{\mathcal{J}} = D^{\mathcal{I}}$. Since $\mathcal{I} \models \mathcal{O}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and hence $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$. Therefore, $\mathcal{J}$ satisfies the axiom.

2. $g(C) = T$ and $g(D) = F$. In this case $C^{\mathcal{J}} = C^{\mathcal{I}}$ and $D^{\mathcal{J}} = D^{\mathcal{I}} \cup \mathbf{S}$. Since $\mathcal{I} \models \mathcal{O}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and hence $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$. Therefore, again $\mathcal{J}$ satisfies the axiom.

3. $g(C) = F$ and $g(D) = F$. In this case $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \mathbf{S}$ and $D^{\mathcal{J}} = D^{\mathcal{I}} \cup \mathbf{S}$. Since $\mathcal{I} \models \mathcal{O}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and hence $C^{\mathcal{I}} \cup \mathbf{S} \subseteq D^{\mathcal{I}} \cup \mathbf{S}$. Therefore, $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$ and again $\mathcal{J}$ satisfies the axiom.

For axioms of the form $R \sqsubseteq S$, for R, S roles, since $R^{\mathcal{I}} \sqsubseteq S^{\mathcal{I}}$ and $R^{\mathcal{I}} = R^{\mathcal{J}}$ and $S^{\mathcal{I}} = S^{\mathcal{J}}$ for any $\mathcal{SHOIN}$ role (atomic or not) , we have that $\mathcal{J}$ satisfies the axiom.

For axioms of the form $C(a)$, since $a^{\mathcal{I}} \in C^{\mathcal{I}}$, and $a^{\mathcal{I}} = a^{\mathcal{J}}$ and $C^{\mathcal{J}}$ is either $C^{\mathcal{I}}$ (if $g(C) = T$) or $C^{\mathcal{I}} \cup \mathbf{S}$, if $g(C) = F$, we have that in both cases $a^{\mathcal{J}} \in C^{\mathcal{J}}$. Similar arguments can be used to verify that if $\mathcal{I}$ satisfies an axiom of the form $R(a, b)$ or $Trans(R)$, then $\mathcal{J}$ satisfies it.

Finally, if $\mathcal{O}$ is not safe, then there is a GCI $C \sqsubseteq D$ in $\mathcal{O}$ such that $g(C) = F$ and $g(D) = T$. Since $g(C) = F$, then $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \mathbf{S}$. Since $g(D) = T$, then $D^{\mathcal{J}} = D^{\mathcal{I}}$. Since $\mathbf{S} \neq \emptyset$ and $\mathbf{S} \cap D^{\mathcal{I}} = \emptyset$, then $\mathcal{J}$ does not satisfy the GCI and hence $\mathcal{J}$ is not a model of $\mathcal{O}$.

Q.E.D

The lemma shows that the class of models of an safe ontology is closed under domain expansions. We are now ready to prove Theorem 1.

Theorem 1 *Let* $\mathcal{O}$ *with vocabulary* V *be consistent and safe, then there are no* $\mathcal{SHOIN}$ *concepts C,D in the vocabulary* V *s.t.* $g(C) = F$*,* $g(D) = T$ *and* $\mathcal{O} \models (C \sqsubseteq D)$*.*

Proof

Let $\mathcal{O}$ with vocabulary V be safe. We prove that there is no axiom of the form $C \sqsubseteq D$ with $g(C) = F$ and $g(D) = T$ such that $\mathcal{O} \models (C \sqsubseteq D)$.

Let $\mathcal{I} = (\mathcal{W}_I, .^{\mathcal{I}})$ be an interpretation such that $\mathcal{I} \models \mathcal{O}$ and $\mathcal{I} \models (C \sqsubseteq D)$. Let $\mathcal{J} = (\mathcal{W}_J, .^{\mathcal{J}})$ the interpretation obtained from $\mathcal{I}$ by a domain expansion with set $\mathbf{S}$.

By Lemma 3, $\mathcal{J}$ is a model of $\mathcal{O}$. However, we show that it does not satisfy the axiom $C \sqsubseteq D$.

Since $g(C) = F$, then $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \mathbf{S}$ and since $g(D) = T$ we have that $D^{\mathcal{J}} = D^{\mathcal{I}}$. Now we have that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. However $\mathbf{S} \cap D^{\mathcal{I}} = \emptyset$ by construction and hence $C^{\mathcal{J}}$ is not a subset of $D^{\mathcal{J}}$. This implies that $\mathcal{O}$ cannot entail such an axiom, since there are models of $\mathcal{O}$ that do not satisfy it. **Q.E.D**

A.2 Proof for Theorems 2 and 3

Lemma 4 *Let* $G(\mathcal{O}) = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$ *with* $|\mathbf{V}| = n$*. Let* $\mathcal{J} = (\mathcal{W}_J, .^{\mathcal{J}})$ *be an arbitrary interpretation for* $\mathcal{O}$ *of the following form:*

- $\mathcal{W}_J = \bigcup_{i=1,...,n} \mathcal{W}_i$ *with* $\mathcal{W}_i \cap \mathcal{W}_j = \emptyset$ *for* $i \neq j$*, and* $\mathcal{W}_i \neq \emptyset$
- $A^{\mathcal{J}} \subseteq \mathcal{W}_i$*, for each* $A \in V_C$ *with* $\mathcal{V}(A) = v_i$
- $R^{\mathcal{J}} \subseteq \mathcal{W}_i \times \mathcal{W}_j$*, for each* $R \in V_R$ *with* $\mathcal{V}(R) = (v_i, v_j)$*.*
- $a^{\mathcal{J}} \in \mathcal{W}_i$ *for each* $a \in V_I$ *with* $\mathcal{V}(a) = v_i$

Let C *be a concept occurring in* $\mathcal{O}$*. Let* $\mathcal{V}(C) = v_i$ *and* $x \notin \mathcal{W}_i$*, then:*

- *If* $g(C) = T$*, then* $x \notin C^{\mathcal{J}}$
- *If* $g(C) = F$*, then* $x \in C^{\mathcal{J}}$

Proof We proceed by induction. At the base of the induction, we have:

- If C is an atomic concept, $g(C) = T$. By definition, $C^{\mathcal{J}} \subseteq \mathcal{W}_i$. Since, $\mathcal{W}_i \cap \mathcal{W}_j = \emptyset$ for any $i \neq j$, the claim holds.
- If C is of the form $\{a\}$, $g(C) = T$. We have by construction of $G(\mathcal{O})$ that $\mathcal{V}(a) = v_i$. Furthermore, $C^{\mathcal{J}} = \{a^{\mathcal{J}}\}$ with $a^{\mathcal{J}} \in \mathcal{W}_i$. Again, the claim holds.
- If C is of the form $\geq nR$, $g(C) = T$. By construction of $G(\mathcal{O})$, $\mathcal{V}(R) = (v_i, v_j)$ for some $j \in \{1, ..., n\}$. By definition of $\mathcal{J}$, $R^{\mathcal{J}} \subseteq \mathcal{W}_i \times \mathcal{W}_j$ and hence if $x \notin \mathcal{W}_i$, then $x \notin (\geq nR)^{\mathcal{J}}$
- If C is of the form $\leq nR$, $g(C) = F$. By construction of $G(\mathcal{O})$, $\mathcal{V}(R) = (v_i, v_j)$ for some $j \in \{1, ..., n\}$. Again, by definition of $\mathcal{J}$, $R^{\mathcal{J}} \subseteq \mathcal{W}_i \times \mathcal{W}_j$. If $x \notin \mathcal{W}_i$, it follows that there is no $y \in \mathcal{W}_J$ s.t. $(x, y) \in R^{\mathcal{J}}$. Hence, $x \in (\leq nR)^{\mathcal{J}}$
- If C is of the form $\exists R.D$, then $g(C) = T$. By construction of $G(\mathcal{O})$, we have that $\mathcal{V}(R) = (v_i, v_j)$ and $\mathcal{V}(D) = v_j$, for some $j \in \{1, ..., n\}$. Also, $R^{\mathcal{J}} \subseteq \mathcal{W}_i \times \mathcal{W}_j$. Hence, if $x \notin \mathcal{W}_i$, then there is no $y \in \mathcal{W}_J$ s.t. $(x, y) \in R^{\mathcal{J}}$ and therefore $x \notin C^{\mathcal{J}}$
- If C is of the form $\forall R.D$, $g(C) = F$. By construction of $G(\mathcal{O})$, $\mathcal{V}(R) = (v_i, v_j)$ for some $j \in \{1, ..., n\}$. Again, by definition of $\mathcal{J}$, $R^{\mathcal{J}} \subseteq \mathcal{W}_i \times \mathcal{W}_j$. If $x \notin \mathcal{W}_i$, it follows that there is no $y \in \mathcal{W}_J$ s.t. $(x, y) \in R^{\mathcal{J}}$. Hence, $x \in (\forall R.D)^{\mathcal{J}}$

We verify the induction step:

- If C is of the form $\neg D$, we have by construction of $G(\mathcal{O})$ that $g(C) = g(D) = v_i$. By the semantics, $C^{\mathcal{J}} = \mathcal{W}_J - D^{\mathcal{J}}$. Let $x \notin \mathcal{W}_i$. Two possibilities:
  - $g(D) = T$ and $g(C) = F$. If $g(D) = T$, by induction hypothesis, $x \notin D^{\mathcal{J}}$ and hence $x \in \mathcal{W}_J - D^{\mathcal{J}}$
  - $g(D) = F$ and $g(C) = T$. If $g(D) = F$, by induction hypothesis, $x \in D^{\mathcal{J}}$. Hence, $x \notin \mathcal{W}_J - D^{\mathcal{J}}$.
- If C is of the form $D \sqcap E$, we have by construction of $G(\mathcal{O})$ that $g(C) = g(D) = g(E) = v_i$. By the semantics, $C^{\mathcal{J}} = D^{\mathcal{J}} \cap E^{\mathcal{J}}$. Let $x \notin \mathcal{W}_i$. Two possibilities:
  - If $g(C) = F$, then $g(D) = F$ and $g(E) = F$. By induction hypothesis, $x \in D^{\mathcal{J}}$ and $x \in E^{\mathcal{J}}$. Hence, $x \in C^{\mathcal{J}}$
  - If $g(C) = T$, then either $g(D) = T$ or $g(E) = T$. By induction hypothesis this means that either $x \notin D^{\mathcal{J}}$ or $x \notin E^{\mathcal{J}}$. Hence, $x \notin C^{\mathcal{J}}$
- If C is of the form $D \sqcup E$, we have by construction of $G(\mathcal{O})$ that $g(C) = g(D) = g(E) = v_i$. By definition, $C^{\mathcal{J}} = D^{\mathcal{J}} \cup E^{\mathcal{J}}$. Let $x \notin \mathcal{W}_i$. Two possibilities:
  - If $g(C) = T$, then $g(D) = T$ and $g(E) = T$. By induction hypothesis, $x \notin D^{\mathcal{J}}$ and $x \notin E^{\mathcal{J}}$. Hence, $x \notin C^{\mathcal{J}}$
  - If $g(C) = F$, then either $g(D) = F$ or $g(E) = F$. By induction hypothesis this means that either $x \in D^{\mathcal{J}}$ or $x \in E^{\mathcal{J}}$. Hence, $x \in C^{\mathcal{J}}$

Q.E.D

Theorem 2 *Let $\mathcal{O}$ be consistent and safe and $G(\mathcal{O}) = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$ with $|\mathbf{V}| = n$, then there exists a model $\mathcal{J} = (\mathcal{W}_J, .^{\mathcal{J}})$ of $\mathcal{O}$ such that:*

$\mathcal{W}_J = \bigcup_{i=1,\ldots,n} \mathcal{W}_i$ *with* $\mathcal{W}_i \cap \mathcal{W}_j = \emptyset$ *for* $i \neq j$, *and* $\mathcal{W}_i \neq \emptyset$

$A^{\mathcal{J}} \subseteq \mathcal{W}_i$, *for each* $A \in V_C$ *with* $\mathcal{V}(A) = v_i$

$R^{\mathcal{J}} \subseteq \mathcal{W}_i \times \mathcal{W}_j$, *for each* $R \in V_R$ *with* $\mathcal{V}(R) = (v_i, v_j)$.

$a^{\mathcal{J}} \in \mathcal{W}_i$ *for each* $a \in V_I$ *with* $\mathcal{V}(a) = v_i$

Proof

Since $\mathcal{O}$ is consistent, there exists an interpretation $\mathcal{I} = (\mathcal{W}, .^{\mathcal{I}})$ s.t. $\mathcal{I} \models \mathcal{O}$. We show that we can obtain from $\mathcal{I}$ an interpretation $\mathcal{J}$ of the desired form s.t. $\mathcal{J} \models \mathcal{O}$.

First, we define the domain $\mathcal{W}_J$ of $\mathcal{J}$ using the following steps:

1. $\mathcal{W}_J \leftarrow \emptyset$

2. For every $x \in \mathcal{W}_I$, generate n new objects $x_1, ..., x_n$ and do $\mathcal{W}_J \leftarrow \mathcal{W}_J \cup \{x_1, ..., x_n\}$.

Now, we define the interpretation function $.^{\mathcal{J}}$ as well as the n sets $(\mathcal{W}_i)_{1 \leq i \leq n}$ as follows:

1. Initialize $\mathcal{W}_i \leftarrow \emptyset$ for all $1 \leq i \leq n$

2. For every individual $a \in V_I$ with $\mathcal{V}(a) = v_i$ with $i \in \{1, ..., n\}$, do $a^{\mathcal{J}} = a_i$; $\mathcal{W}_i \leftarrow \mathcal{W}_i \cup \{a_i\}$.

3. Initialize $A^{\mathcal{J}} \leftarrow \emptyset; R^{\mathcal{J}} \leftarrow \emptyset$ for every atomic concept $A \in V_C$ and atomic role $R \in V_R$.

4. For every atomic concept $A \in V_C$ with $\mathcal{V}(A) = v_i$ for some $i \in \{1, ..., n\}$ and every $x \in A^{\mathcal{I}}$, do $A^{\mathcal{J}} \leftarrow A^{\mathcal{J}} \cup \{x_i\}$ and $\mathcal{W}_i \leftarrow \mathcal{W}_i \cup \{x_i\}$.

5. For every atomic role $R \in V_R$ with $\mathcal{V}(R) = (v_i, v_j)$ for some $i, j \in \{1, ..., n\}$ and every pair $(x, y) \in R^{\mathcal{I}}$, do $R^{\mathcal{J}} \leftarrow R^{\mathcal{J}} \cup (x_i, y_j)$, $\mathcal{W}_i \leftarrow \mathcal{W}_i \cup \{x_i\}; \mathcal{W}_j \leftarrow \mathcal{W}_j \cup \{y_j\}$

By construction, it is easy to see that:

- $\mathcal{W}_J = \bigcup_{i=1,...,n} \mathcal{W}_i$, with $\mathcal{W}_i \neq \emptyset$ for $1 \leq i \leq n$
- $\mathcal{W}_i \cap \mathcal{W}_j = \emptyset$ for $i \neq j$
- $A^{\mathcal{J}} \subseteq \mathcal{W}_i$, for each $A \in V_C$ with $\mathcal{V}(A) = v_i$
- $R^{\mathcal{J}} \subseteq \mathcal{W}_i \times \mathcal{W}_j$, for each $R \in V_R$ with $\mathcal{V}(R) = (v_i, v_j)$.
- $a^{\mathcal{J}} \in \mathcal{W}_i$ for each $a \in V_I$ with $\mathcal{V}(a) = v_i$

Hence, $\mathcal{J}$ verifies the conditions of Lemma 4. Note also that, by construction of $\mathcal{J}$:

$$(x_i, y_j) \in R^{\mathcal{J}} \Leftrightarrow (x, y) \in R^{\mathcal{I}}$$

for every role (atomic or not).

We show that $\mathcal{J} \models \mathcal{O}$. For such a purpose, we start by proving the following (♣):

CLAIM:(♣)

Let C be a concept occurring in $\mathcal{O}$, with $\mathcal{V}(C) = v_i$, for some $i \in \{1, ..., n\}$. The following holds:

$$\text{If } x_i \in \mathcal{W}_i \text{, then } x_i \in C^{\mathcal{J}} \Leftrightarrow x \in C^{\mathcal{I}}$$

Proof for ♣:
We proceed by induction. At the base of the induction we have:

- Let C an atomic concept with $\mathcal{V}(C) = v_i$. By definition of $\mathcal{J}$, ♣ holds.
- If C is of the form $\{a\}$, by construction of $G(\mathcal{O})$, $\mathcal{V}(C) = \mathcal{V}(a)$. By definition of $\mathcal{J}$, $a^{\mathcal{J}} = a_i$ with $a_i \in \mathcal{W}_i$. Hence, $C^{\mathcal{J}} = \{a_i\}$. On the other hand, $C^{\mathcal{I}} = \{a\}$, which implies that ♣ holds.
- C of the form $\geq nR$, with $\mathcal{V}(R) = (v_i, v_j)$ for some $j \in \{1, ..., n\}$. Let $x_i \in \mathcal{W}_i$. We prove the two directions of ♣.
  - ($\Rightarrow$) Let $x_i \in (\geq nR)^{\mathcal{J}}$. By the semantics, $||\{(x_i, y_j) \in R^{\mathcal{J}}$, with $y_j \in \mathcal{W}_J\}|| \geq n\}$. On the other hand, $(x_i, y_j) \in R^{\mathcal{J}} \Leftrightarrow (x, y) \in R^{\mathcal{I}}$, with $x, y \in \mathcal{W}_I$. Hence, $||\{(x, y) \in R^{\mathcal{I}}$ with $y \in \mathcal{W}_I\}|| \geq n$ and thus $x \in C^{\mathcal{I}}$.
  - ($\Leftarrow$) Let $x \in C^{\mathcal{I}}$, then $||\{(x, y) \in R^{\mathcal{I}}$ with $y \in \mathcal{W}_I\}|| \geq n$. Again, it holds that $(x, y) \in R^{\mathcal{I}} \Leftrightarrow (x_i, y_j) \in R^{\mathcal{J}}$. Hence, $||\{(x_i, y_j) \in R^{\mathcal{J}}$ with $y_j \in \mathcal{W}_J\}|| \geq n$ and $x_i \in C^{\mathcal{J}}$.
- Let C be of the form $\leq nR$, with $\mathcal{V}(R) = (v_i, v_j)$ for some $i, j \in \{1, ..., n\}$. Let $x_i \in \mathcal{W}_i$. We prove the two directions of ♣.
  - ($\Rightarrow$) Let $x_i \in C^{\mathcal{J}}$. We have that $(x_i, y_j) \in R^{\mathcal{J}} \Leftrightarrow (x, y) \in R^{\mathcal{I}}$, with $x, y \in \mathcal{W}_I$, $x_i \in \mathcal{W}_i$ and $y_j \in \mathcal{W}_j$. By the semantics there are at most n elements $y_j \in \mathcal{W}_j$ s.t. $(x_i, y_j) \in R^{\mathcal{J}}$. Hence, there are at most n elements $y \in \mathcal{W}_I$ s.t. $(x, y) \in R^{\mathcal{I}}$. Consequently, $x \in (\leq nR)^{\mathcal{I}}$
  - ($\Leftarrow$). Let $x_i \in C^{\mathcal{I}}$. Again, it holds that $(x_i, y_j) \in R^{\mathcal{J}} \Leftrightarrow (x, y) \in R^{\mathcal{I}}$. By the semantics, there are at most n elements $y \in \mathcal{W}_I$ s.t. $(x, y) \in R^{\mathcal{I}}$ and hence there are also at most n elements $y_j \in \mathcal{W}_j$ s.t. $(x_i, y_j) \in R^{\mathcal{J}}$. Hence, $x_i \in (\leq nR)^{\mathcal{J}}$

We take ♣ as induction hypothesis.

- C of the form $\neg D$. If $\mathcal{V}(D) = v_i$, then $\mathcal{V}(C) = v_i$, by construction of $G(\mathcal{O})$. By the semantics, $C^{\mathcal{J}} = \mathcal{W}_J - D^{\mathcal{J}}$ and $C^{\mathcal{I}} = \mathcal{W}_I - D^{\mathcal{I}}$. Let $x_i \in \mathcal{W}_i$, we prove the two directions of ♣:
  - ($\Rightarrow$) Let $x_i \in C^{\mathcal{J}}$, then $x_i \notin D^{\mathcal{J}}$. By induction hypothesis, $x \notin D^{\mathcal{I}}$ and hence $x \in C^{\mathcal{I}}$.
  - Let now $x \in C^{\mathcal{I}}$, then by definition $x \notin D^{\mathcal{I}}$. By induction hypothesis $x_i \notin D^{\mathcal{J}}$ and hence $x_i \in C^{\mathcal{J}}$.

- C of the form $D \sqcap E$. If $\mathcal{V}(C) = v_i$, then by construction of $G(\mathcal{O})$, $\mathcal{V}(D) = \mathcal{V}(E) = v_i$. By the semantics, $C^{\mathcal{J}} = D^{\mathcal{J}} \cap E^{\mathcal{J}}$ and $C^{\mathcal{I}} = D^{\mathcal{I}} \cap E^{\mathcal{I}}$. Let $x_i \in \mathcal{W}_i$, we prove the two directions of $\clubsuit$:
  - ($\Rightarrow$) Let $x_i \in C^{\mathcal{J}}$, then $x_i \in D^{\mathcal{J}}$ and $x_i \in E^{\mathcal{J}}$. By induction hypothesis, $x \in D^{\mathcal{I}}$ and $x \in E^{\mathcal{I}}$ and hence $x \in C^{\mathcal{I}}$.
  - Let now $x \in C^{\mathcal{I}}$, then by definition $x \in D^{\mathcal{I}}$ and $x \in E^{\mathcal{I}}$. By induction hypothesis, $x_i \in D^{\mathcal{J}}$ and $x_i \in E^{\mathcal{J}}$ and hence $x_i \in C^{\mathcal{J}}$
- C of the form $D \sqcup E$. If $\mathcal{V}(C) = v_i$, then by construction of $G(\mathcal{O})$, $\mathcal{V}(D) = \mathcal{V}(E) = v_i$. By the semantics, $C^{\mathcal{J}} = D^{\mathcal{J}} \cup E^{\mathcal{J}}$ and $C^{\mathcal{I}} = D^{\mathcal{I}} \cup E^{\mathcal{I}}$. Let $x_i \in \mathcal{W}_i$, we prove the two directions of $\clubsuit$:
  - ($\Rightarrow$) Let $x_i \in C^{\mathcal{J}}$, then $x_i \in D^{\mathcal{J}}$ or $x_i \in E^{\mathcal{J}}$. Suppose that $x_i \in D^{\mathcal{J}}$, then by induction hypothesis, $x \in D^{\mathcal{I}}$ and hence $x \in C^{\mathcal{I}}$. If $x_i \in E^{\mathcal{J}}$, then by induction hypothesis, $x \in E^{\mathcal{I}}$ and again $x \in C^{\mathcal{I}}$
  - Let now $x \in C^{\mathcal{I}}$, then by definition $x \in D^{\mathcal{I}}$ or $x \in E^{\mathcal{I}}$. If $x \in D^{\mathcal{I}}$, then by induction hypothesis, $x_i \in D^{\mathcal{J}}$ and hence $x_i \in C^{\mathcal{J}}$. If $x \in E^{\mathcal{I}}$, then by induction hypothesis, $x_i \in E^{\mathcal{J}}$ and again $x_j \in C^{\mathcal{J}}$.
- C of the form $\exists R.D$. If $\mathcal{V}(R) = (v_i, v_j)$, then by construction of $G(\mathcal{O})$, $\mathcal{V}(C) = v_i$ and $\mathcal{V}(D) = v_j$. Let $x_i \in \mathcal{W}_i$, we prove the two directions of $\clubsuit$:
  - ($\Rightarrow$) If $x_i \in C^{\mathcal{J}}$, then there exists an element $y_j \in \mathcal{W}_j$ s.t. $(x_i, y_j) \in R^{\mathcal{J}}$ and $y_j \in D^{\mathcal{J}}$. We have that $(x_i, y_j) \in R^{\mathcal{J}} \Leftrightarrow (x, y) \in R^{\mathcal{I}}$ and hence $(x, y) \in R^{\mathcal{I}}$. By induction hypothesis, $y \in D^{\mathcal{I}}$. Therefore $x \in C^{\mathcal{I}}$
  - ($\Leftarrow$) If $x \in C^{\mathcal{I}}$, then there exists an element $y \in \mathcal{W}_I$ s.t. $(x, y) \in R^{\mathcal{I}}$ and $y \in D^{\mathcal{I}}$. We have that $(x_i, y_j) \in R^{\mathcal{J}} \Leftrightarrow (x, y) \in R^{\mathcal{I}}$ and hence $(x_i, y_j) \in R^{\mathcal{J}}$. By induction hypothesis, $y_j \in D^{\mathcal{J}}$ and thus $x_i \in C^{\mathcal{J}}$
- C of the form $\forall R.D$. If $\mathcal{V}(R) = (v_i, v_j)$, then by construction of $G(\mathcal{O})$, $\mathcal{V}(C) = v_i$ and $\mathcal{V}(D) = v_j$. Let $x_i \in \mathcal{W}_i$, we prove the two directions of $\clubsuit$:
  - ($\Rightarrow$) If $x_i \in C^{\mathcal{J}}$, then if $(x_i, y_j) \in R^{\mathcal{J}}$ necessarily $y_j \in D^{\mathcal{J}}$. We have that $(x_i, y_j) \in R^{\mathcal{J}} \Leftrightarrow (x, y) \in R^{\mathcal{I}}$ and hence $(x, y) \in R^{\mathcal{I}}$. By induction hypothesis, $y \in D^{\mathcal{I}}$. Therefore $x \in C^{\mathcal{I}}$
  - ($\Leftarrow$) If $x \in C^{\mathcal{I}}$, then if $(x, y) \in R^{\mathcal{I}}$ necessarily $y \in D^{\mathcal{I}}$. We have that $(x_i, y_j) \in R^{\mathcal{J}} \Leftrightarrow (x, y) \in R^{\mathcal{I}}$ and hence $(x_i, y_j) \in R^{\mathcal{J}}$. By induction hypothesis, $y_j \in D^{\mathcal{J}}$ and thus $x_i \in C^{\mathcal{J}}$

Now we show that $\mathcal{J}$ satisfies every axiom in $\mathcal{O}$. We start with GCIs. Let $C \sqsubseteq D$ be in $\mathcal{O}$. By construction of $G(\mathcal{O})$, we have that $\mathcal{V}(C) = \mathcal{V}(D) = v_i$ for some $i \in \{1, ..., n\}$. Since $\mathcal{O}$ is safe, we have three possibilities:

- $g(C) = g(D) = T$. We show that $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$. Let $x_i \in C^{\mathcal{J}}$. Since $g(C) = T$, we have by Lemma 4 that $x_i \in \mathcal{W}_i$.By $\clubsuit$, $x \in C^{\mathcal{I}}$. Since $\mathcal{I} \models \mathcal{O}$, we have that $x \in D^{\mathcal{I}}$. By $\clubsuit$, $x_i \in D^{\mathcal{J}}$ and hence $\mathcal{J} \models (C \sqsubseteq D)$.

- $g(C) = T$ and $g(D) = F$. We show that $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$. Let $x_i \in C^{\mathcal{J}}$. Since $g(C) = T$, we have by Lemma 4 that $x_i \in \mathcal{W}_i$, then by $\clubsuit$, $x \in D^{\mathcal{I}}$. Since $\mathcal{I} \models \mathcal{O}$, we have that $x \in D^{\mathcal{I}}$. By $\clubsuit$, $x_i \in D^{\mathcal{J}}$ and hence again $\mathcal{J} \models (C \sqsubseteq D)$.

- $g(C) = F$ and $g(D) = F$. We show that $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$. Let $x_i \in C^{\mathcal{J}}$, then we have two possibilities:
  - $x_i \in \mathcal{W}_i$. By $\clubsuit$, $x \in C^{\mathcal{I}}$. Since $\mathcal{I} \models \mathcal{O}$, we have that $x \in D^{\mathcal{I}}$. By $\clubsuit$, $x_i \in D^{\mathcal{J}}$ and hence $\mathcal{J} \models (C \sqsubseteq D)$.
  - $x_i \notin \mathcal{W}_i$. By Lemma 4, since $g(D) = F$ and $\mathcal{V}(D) = v_i$, we have that $x_i \in D^{\mathcal{J}}$

Let $C(a) \in \mathcal{O}$. By construction of $G(\mathcal{O})$, $\mathcal{V}(C) = \mathcal{V}(a) = v_i$ for some $i \in \{1, ..., n\}$. Since $\mathcal{I} \models C(a)$, we have that $a \in C^{\mathcal{I}}$. Also $a_i \in \mathcal{W}_i$. By $\clubsuit$, $a_i \in \mathcal{C}^{\mathcal{J}}$, i.e. $a^{\mathcal{J}} \in C^{\mathcal{J}}$ and hence $\mathcal{J} \models C(a)$.

The remaining cases: $R(a, b) \in \mathcal{O}$, $R \sqsubseteq S \in \mathcal{O}$, $Trans(R) \in \mathcal{O}$ are straightforward to verify. Just remember that, by construction of $G(\mathcal{O})$, if $\mathcal{V}(R) = (v_i, v_j)$, then $\mathcal{V}(a) = v_i$, $\mathcal{V}(b) = v_j$ and $\mathcal{V}(S) = (v_i, v_j)$.

Q.E.D

Theorem 3 *Given $G(\mathcal{O})$, the axioms of the following form* ***cannot*** *be entailed by a consistent ontology $\mathcal{O}$:*

- *$C \sqsubseteq D$, with C, D concepts s.t. $g(C) = g(D) = T$ and $\mathcal{V}(A) \neq \mathcal{V}(B)$, provided that C is satisfiable.*
- *$C(a)$ with C a concept s.t. $g(C) = T$ and $\mathcal{V}(a) \neq \mathcal{V}(C)$.*
- *$R \sqsubseteq S$ with R, S roles and $\mathcal{V}(R) \neq \mathcal{V}(S)$.*
- *$R(a, b)$ with $\mathcal{V}(R) \neq (\mathcal{V}(a), \mathcal{V}(b))$*

Proof

By Theorem 2, there exists a model $\mathcal{I} = (\mathcal{W}, .^{\mathcal{I}})$ of $\mathcal{O}$ such that:

- $\mathcal{W} = \bigcup_{i=1,...,n} \mathcal{W}_i$ with $\mathcal{W}_i \cap \mathcal{W}_j = \emptyset$ for $i \neq j$, and $\mathcal{W}_i \neq \emptyset$
- $A^{\mathcal{I}} \subseteq \mathcal{W}_i$, for each $A \in V_C$ with $\mathcal{V}(A) = v_i$
- $R^{\mathcal{I}} \subseteq \mathcal{W}_i \times \mathcal{W}_j$, for each $R \in V_R$ with $\mathcal{V}(R) = (v_i, v_j)$.

```
-Algorithm Merge($v, w, G$)
-Input: A partitioning graph $G = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$ for $\mathcal{O}$
    Two nodes $v, w$ in $G$
-Output: A partitioning graph $G' = (\mathbf{V}', \mathbf{E}', \mathcal{L}', \mathcal{V}')$

- $G' \leftarrow G$
- $\mathcal{L}'(v) \leftarrow \mathcal{L}(v) \cup \mathcal{L}(w)$
- $\mathcal{V}'(X) \leftarrow v$ for every atomic concept or individual $X$ s.t. $\mathcal{V}(X) = w$
- for each edge $(u, w) \in \mathbf{E}$, do:
    $\mathbf{E}' \leftarrow \mathbf{E}' \cup (u, v)$
    $\mathcal{L}'((u, v)) \leftarrow \mathcal{L}((u, w))$
    $\mathbf{E}' \leftarrow \mathbf{E}' - (u, w)$
- for each edge $(w, z) \in \mathbf{E}$, do:
    $\mathbf{E}' \leftarrow \mathbf{E}' \cup (v, z)$
    $\mathcal{L}'((v, z)) \leftarrow \mathcal{L}'((w, z))$
    $\mathbf{E}' \leftarrow \mathbf{E}' - (w, z)$
return $G'$
```

Figure 8: Merging two nodes in a partitioning graph

- $a^{\mathcal{I}} \in \mathcal{W}_i$ for each $a \in V_I$ with $\mathcal{V}(a) = v_i$

Suppose $\mathcal{O} \models C \sqsubseteq D$ with $\mathcal{V}(C) \neq \mathcal{V}(D)$. Since $\mathcal{O} \models C \sqsubseteq D$, then every model of $\mathcal{O}$ satisfies the axiom $C \sqsubseteq D$. In particular $\mathcal{I} \models C \sqsubseteq D$. Since C is satisfiable in $\mathcal{O}$, there exists an element $x \in \mathcal{W}$ s.t. $x \in C^{\mathcal{I}}$. By Lemma 4, $x \in \mathcal{W}_i$. Also by Lemma 4, since $g(D) = T$, $D^{\mathcal{I}} \subseteq \mathcal{W}_j$. Since $\mathcal{W}_i \cap \mathcal{W}_j = \emptyset$, $x \notin D^{\mathcal{I}}$ and hence $\mathcal{I}$ does not satisfy the axiom, which contradicts our assumption.

Suppose $\mathcal{O} \models C(a)$ with $\mathcal{V}(a) \neq \mathcal{V}(C)$. Again, since $\mathcal{O} \models C(a)$, then $\mathcal{I} \models C(a)$. However, since $\mathcal{V}(C) \neq \mathcal{V}(a)$, $a^{\mathcal{I}} \notin \mathcal{W}_i$ and by Lemma 4, $a^{\mathcal{I}} \notin C^{\mathcal{I}}$. Hence $\mathcal{I}$ does not satisfy the axiom.

Suppose $\mathcal{O} \models R \sqsubseteq S$ with $\mathcal{V}(R) \neq \mathcal{V}(S)$. Again, the assumption implies that $\mathcal{I} \models R \sqsubseteq S$. However, since $\mathcal{V}(R) \neq \mathcal{V}(S)$, $R^{\mathcal{I}}$ and $S^{\mathcal{I}}$ are disjoint and hence $\mathcal{I}$ does not satisfy the axiom.

Suppose $\mathcal{O} \models R(a, b)$ with $R(a, b)$ with $\mathcal{V}(R) \neq (\mathcal{V}(a), \mathcal{V}(b))$. Then, $\mathcal{I} \models R(a, b)$. However, according to the definition of $\mathcal{I}$, this only happens if $\mathcal{V}(R) = (\mathcal{V}(a), \mathcal{V}(b))$, which contradicts the assumption.

Q.E.D

A.3 Proof for Theorem 4

Definition 5 *Let $G = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$ be a partitioning graph for $\mathcal{O}$. Let $v, w \in \mathbf{V}$. The operation $Merge(v, w, G)$ returns a new graph $G' = (\mathbf{V}', \mathbf{E}', \mathcal{L}', \mathcal{V}')$ as given in Figure 8.*

Clearly the graph obtained through a merge is also a partitioning graph for $\mathcal{O}$.

Definition 6 *Let $G = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$ be a partitioning graph for $\mathcal{O}$ and $\mathbf{S} \subseteq \mathbf{V}$.*

```
-Algorithm Collapse$(G, \mathbf{S})$
-Input: A partitioning graph $G = (\mathbf{V}, \mathbf{E}, \mathcal{L}, \mathcal{V})$ for $\mathcal{O}$
    A set of nodes $\mathbf{S} \subseteq \mathbf{V}$
-Output: A partitioning graph $G' = (\mathbf{V}', \mathbf{E}', \mathcal{L}', \mathcal{V}')$

- $G' \leftarrow G$
- $v \leftarrow$ Arbitrary element of $\mathbf{S}$
- for each $w \in \mathbf{S}$, with $w \neq v$ do
    $G' \leftarrow Merge(G', v, w)$.
- if $||\mathbf{V}'|| \leq 2$, then return $G'$
- else
    $\mathbf{S} \leftarrow \mathbf{V}' - \{v\}$
    $u \leftarrow$ Arbitrary node in $\mathbf{S}$
    for each $w \in \mathbf{S}$ with $w \neq u$ do
        $G' \leftarrow Merge(G', u, w)$
return $G'$
```

Figure 9: Collapsing nodes

The operation $Collapse(G, \mathbf{S})$ returns a new graph $G' = (\mathbf{V}', \mathbf{E}', \mathcal{L}', \mathcal{V}')$ as given in Figure 9.

It is not hard to see that, if $G = Partition(\mathcal{O})$ and Γ the module for X in G $\Gamma = \mathcal{L}(v_1) \cup ...\mathcal{L}(v_k)$, then the graph $G' = Collapse(G, \mathbf{S})$, with $\mathbf{S} = \{v_1, ..., v_k\}$ is of either of the following forms:

- It contains a single node v with $\mathcal{L}(v) = \mathcal{O}$.
- It contains two nodes v_1, v_2 s.t. $\mathcal{L}(v_2) = \Gamma$ and either of the following holds:
    - v_1 and v_2 are disconnected ($(v_1, v_2) \notin \mathbf{V}$ and $(v_2, v_1) \notin \mathbf{V}$)
    - v_1 and v_2 are only connected by an edge (v_1, v_2)

We will use this fact to prove Theorem 4.

Lemma 5 *Let G be a partitioning graph for $\mathcal{O}$ of the form $G = (\{v_1, v_2\}, \{(v_1, v_1), (v_2, v_2)\}, \mathcal{L}, \mathcal{V})$. Let $\mathcal{I}_1 = (\mathcal{W}_1, .^{\mathcal{I}_1})$ be a model of $\mathcal{L}(v_1)$ and $\mathcal{I}_2 = (\mathcal{W}_2, .^{\mathcal{I}_2})$ a model of $\mathcal{L}(v_2)$, with $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$.*

Then the interpretation $\mathcal{J} = (\mathcal{W}, .^{\mathcal{J}})$, defined as follows:

- $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$
- $A^{\mathcal{J}} = A^{\mathcal{I}_1}$ *if* $\mathcal{V}(A) = v_1$ *and* $A^{\mathcal{J}} = A^{\mathcal{I}_2}$ *if* $\mathcal{V}(A) = v_2$
- $a^{\mathcal{J}} = a^{\mathcal{I}_1}$ *if* $\mathcal{V}(a) = v_1$ *and* $a^{\mathcal{J}} = a^{\mathcal{I}_2}$ *if* $\mathcal{V}(a) = v_2$
- $R^{\mathcal{J}} = R^{\mathcal{I}_1}$ *if* $\mathcal{V}(R) = (v_1, v_1)$ *and* $R^{\mathcal{J}} = R^{\mathcal{I}_2}$ *if* $\mathcal{V}(R) = (v_2, v_2)$

is a model of $\mathcal{O}$ iff $\mathcal{O}$ is safe.

Proof

Let us start by proving the following claim (♠):

CLAIM (♠): Let C be any concept in $\mathcal{O}$ s.t. $\mathcal{V}(C) = v_i$ for some $i \in \{1, 2\}$, then:

$$C^{\mathcal{J}} = C^{\mathcal{I}_i} \text{ if } g(C) = T$$
$$C^{\mathcal{J}} = C^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}} \text{ if } g(C) = F$$

where $\bar{i} = 1$ if $i = 2$ and $\bar{i} = 2$ whenever $i = 1$.

Proof for ♠:

First, note that $R^{\mathcal{J}} = R^{\mathcal{I}_i}$ for every role R (atomic or not) occurring in $\mathcal{O}$ and s.t. $\mathcal{V}(R) = (v_i, v_i)$. The proof for the claim goes by induction on the structure of $\mathcal{SHOIN}$ concepts. At the base of the induction we have:

- If C is an atomic concept, then $g(C) = T$ and, by definition of $\mathcal{J}$, $C^{\mathcal{J}} = C^{\mathcal{I}_i}$.
- If C is $\top$, then $(\top)^{\mathcal{J}} = \mathcal{W} = \mathcal{W}_i \cup \mathcal{W}_{\bar{i}} = (\top)^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$. In this case, $g(C) = F$.
- If C of the form $\{a\}$, for a an individual, then $C^{\mathcal{J}} = \{a^{\mathcal{J}}\}$. By definition, $a^{\mathcal{J}} = a^{\mathcal{I}_i}$, and hence $C^{\mathcal{J}} = C^{\mathcal{I}_i}$. In this case, $g(C) = T$.
- If C of the form $\geq nR$, then $g(C) = T$ and $(\geq nR)^{\mathcal{J}} = \{x \in \mathcal{W} | ||\{y | (x, y) \in R^{\mathcal{J}}\}|| \geq n$. Since $R^{\mathcal{J}} = R^{\mathcal{I}_i}$, it follows that every $x \in (\geq nR)^{\mathcal{J}}$ must belong to $\mathcal{W}_i$. Therefore, it is easy to see that $(\geq nR)^{\mathcal{J}} = (\geq nR)^{\mathcal{I}_i}$.
- If C of the form $\leq nR$, then $g(C) = F$ and $(\leq nR)^{\mathcal{J}} = \{x \in \mathcal{W} | ||\{y | (x, y) \in R^{\mathcal{J}}\}|| \leq n$. Since $R^{\mathcal{J}} = R^{\mathcal{I}_i}$, the set of elements in $\mathcal{W}$ that have at most n elements in $\mathcal{W}$ connected to it by $R^{\mathcal{J}}$ is the union of those that are in $\mathcal{W}_i$ and those in $\mathcal{W}_{\bar{i}}$. The former ones are precisely those in $(\leq nR)^{\mathcal{I}_i}$ and the latter ones are precisely the elements of $\mathcal{W}_{\bar{i}}$, since no element in $\mathcal{W}_{\bar{i}}$ is related through R to an element of the domain. Hence, $(\leq nR)^{\mathcal{J}} = (\leq nR)^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$.

We now take ♠ as induction hypothesis and verify the induction step.

- If C is of the form $\neg D$, then, $C^{\mathcal{J}} = \mathcal{W} - D^{\mathcal{J}}$
  - If $g(D) = T$, then $g(C) = F$. By induction, $D^{\mathcal{J}} = D^{\mathcal{I}_i}$ and $C^{\mathcal{J}} = \mathcal{W} - D^{\mathcal{I}_i}$, which implies that $C^{\mathcal{J}} = (\mathcal{W}_i \cup \mathcal{W}_{\bar{i}}) - D^{\mathcal{I}_i}$. Since $\mathcal{W}_i \cap \mathcal{W}_{\bar{i}} = \emptyset$ for $j \neq i$, then $C^{\mathcal{J}} = (\mathcal{W}_i - D^{\mathcal{I}_i}) \cup \mathcal{W}_{\bar{i}} = C^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$.
  - If $g(D) = F$, then $g(C) = T$. By induction, $D^{\mathcal{J}} = D^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$, and $C^{\mathcal{J}} = \mathcal{W} - (D^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}) = \mathcal{W}_i - D^{\mathcal{I}_i}$. Therefore, $C^{\mathcal{J}} = C^{\mathcal{I}_i}$.
- If C of the form $(D \sqcap E)$, then $(D \sqcap E)^{\mathcal{J}} = D^{\mathcal{J}} \cap E^{\mathcal{J}}$. By induction, $D^{\mathcal{J}}$ is either $D^{\mathcal{I}_i}$ in which case $g(D) = T$ or $D^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$, in which case $g(D) = F$ and analogously, $E^{\mathcal{J}}$ is either $E^{\mathcal{I}_i}$ $(g(E) = T)$ or $E^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$ $(g(E) = F)$. Since

$C^{\mathcal{I}_i}$ and $E^{\mathcal{I}_i}$ are disjoint with $\mathcal{W}_{\bar{i}}$, we have, for every possible combination, that $D^{\mathcal{J}} \cap E^{\mathcal{J}}$ is either equal to $D^{\mathcal{I}_i} \cap E^{\mathcal{I}_i}$ or to $(D^{\mathcal{I}_i} \cap E^{\mathcal{I}_i}) \cup \mathcal{W}_{\bar{i}}$. Note that the latter case only occurs if $D^{\mathcal{J}} = D^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$, and also $E^{\mathcal{J}} = E^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$.

- Let $C = D \sqcup E$. Then $(D \sqcup E)^{\mathcal{J}} = D^{\mathcal{J}} \cup E^{\mathcal{J}}$. Again, by induction, $D^{\mathcal{J}}$ is either $D^{\mathcal{I}_i}$ or $D^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$ and analogously, $E^{\mathcal{J}}$ is either $E^{\mathcal{I}_i}$ or $E^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$. Again, for every possible combination $D^{\mathcal{J}} \cup E^{\mathcal{J}}$ is either equal to $D^{\mathcal{I}_i} \cup E^{\mathcal{I}_i}$ or to $(D^{\mathcal{I}_i} \cup E^{\mathcal{I}_i}) \cup \mathcal{W}_{\bar{i}}$. Note that the former case only occurs if $D^{\mathcal{J}} = D^{\mathcal{I}_i}$ and also $E^{\mathcal{J}} = E^{\mathcal{I}_i}$.

- If C of the form $\exists R.Z$, then $g(C) = T$ and $(\exists R.Z)^{\mathcal{J}} = \{x \in \mathcal{W} | \exists y \in \mathcal{W}, (x, y) \in R^{\mathcal{J}}, y \in Z^{\mathcal{J}}\}$. Note that $R^{\mathcal{J}} = R^{\mathcal{I}_i}$. By induction, we have two cases:
  - Suppose $Z^{\mathcal{J}} = Z^{\mathcal{I}_i}$. In this case, it is easy to see that $(\exists R.Z)^{\mathcal{J}} = (\exists R.Z)^{\mathcal{I}_i}$.
  - Suppose $Z^{\mathcal{J}} = Z^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$. Since $R^{\mathcal{J}} = R^{\mathcal{I}_i}$, there is no element $y \in \mathcal{W}$ s.t. $(x, y) \in R^{\mathcal{J}}$ and $y \in \mathcal{W}_{\bar{i}}$, since y must be in $\mathcal{W}_i$. Therefore $(\exists R.Z)^{\mathcal{J}} = (\exists R.Z)^{\mathcal{I}_i}$.

- If C of the form $\forall R.Z$, then $g(C) = F$ and $(\forall R.Z)^{\mathcal{J}} = \{x \in \mathcal{W} | (x, y) \in R^{\mathcal{J}} \rightarrow y \in Z^{\mathcal{J}}\}$. Again, using that $R^{\mathcal{J}} = R^{\mathcal{I}_i}$ and that $\mathcal{W}_i \cap \mathcal{W}_{\bar{i}} = \emptyset$ it can be seen that $(\forall R.Z)^{\mathcal{J}} = (\forall R.Z)^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$ always holds, independently of the value of $g(Z)$.

Let ϕ be an axiom in $\mathcal{L}(v_i)$. We show that $\mathcal{J} \models \phi$

- Let ϕ be of the form $C \sqsubseteq D$. We have 4 cases:
  1. $g(C) = g(D) = T$. In this case, by $\spadesuit$, $C^{\mathcal{J}} = C^{\mathcal{I}_i}$ and $D^{\mathcal{J}} = D^{\mathcal{I}_i}$. Since $\mathcal{I}_i \models \phi$, $C^{\mathcal{I}_i} \subseteq D^{\mathcal{I}_i}$, and therefore $\mathcal{J} \models \phi$.
  2. $g(C) = T$ and $g(D) = F$. In this case, by $\spadesuit$, $C^{\mathcal{J}} = C^{\mathcal{I}_i}$ and $D^{\mathcal{J}} = D^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$. Since $\mathcal{I}_i$ is a model of $\mathcal{L}(v_i)$, then $\mathcal{I}_i \models \phi$ and hence $C^{\mathcal{I}_i} \subseteq D^{\mathcal{I}_i}$. It follows that $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$ and hence $\mathcal{J} \models \phi$.
  3. $g(C) = F$ and $g(D) = T$. This case only occurs if $\mathcal{O}$ is not safe. We have, by $\spadesuit$, $C^{\mathcal{J}} = C^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$ and $D^{\mathcal{J}} = D^{\mathcal{I}_i}$. In this case, it *always* holds that $C^{\mathcal{J}}$ is not a subset of $D^{\mathcal{J}}$ and hence $\mathcal{J}$ does not satisfy the axiom ϕ, which implies that $\mathcal{J}$ is not a model of $\mathcal{O}$.
  4. $g(C) = g(D) = F$. By $\spadesuit$, $C^{\mathcal{J}} = C^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$ and $D^{\mathcal{J}} = D^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$. Again, since $\mathcal{I}_i \models \phi$, $C^{\mathcal{I}_i} \subseteq D^{\mathcal{I}_i}$, and hence $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$. It follows that $\mathcal{J} \models \phi$.

- Let ϕ of the form $R \sqsubseteq S$. Since $\mathcal{I}_i \models \phi$, then $R^{\mathcal{I}_i} \subseteq S^{\mathcal{I}_i}$ and since $R^{\mathcal{I}_i} = R^{\mathcal{J}}$ and $S^{\mathcal{I}_i} = S^{\mathcal{J}}$, it follows that $R^{\mathcal{J}} \subseteq S^{\mathcal{J}}$. Therefore $\mathcal{J} \models \phi$.

- Let ϕ of the form $C(a)$. We have two cases:

  1. $C^{\mathcal{J}} = C^{\mathcal{I}_i}$. Since $\mathcal{I}_i \models \phi$, then $a^{\mathcal{I}_i} \in C^{\mathcal{I}_i}$. Since $a^{\mathcal{J}} = a^{\mathcal{I}_i}$ and $C^{\mathcal{J}} = C^{\mathcal{I}_i}$, then $a^{\mathcal{J}} \in C^{\mathcal{J}}$ and therefore $\mathcal{J} \models \phi$.
  2. $C^{\mathcal{J}} = C^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$. Since $\mathcal{I}_i \models \phi$, then $a^{\mathcal{I}_i} \in C^{\mathcal{I}_i}$. Since $a^{\mathcal{J}} = a^{\mathcal{I}_i}$ and $C^{\mathcal{J}} = C^{\mathcal{I}_i} \cup \mathcal{W}_{\bar{i}}$, then $a^{\mathcal{J}} \in C^{\mathcal{J}}$ and again $\mathcal{J} \models \phi$.

- Let ϕ of the form $R(a, b)$. Since $\mathcal{I}_i \models \phi$, $(a^{\mathcal{I}_i}, b^{\mathcal{I}_i}) \in R^{\mathcal{I}_i}$. Since $a^{\mathcal{J}} = a^{\mathcal{I}_i}$, $b^{\mathcal{J}} = b^{\mathcal{I}_i}$ and $R^{\mathcal{I}_i} = R^{\mathcal{J}}$, it follows that $(a^{\mathcal{J}}, b^{\mathcal{J}}) \in R^{\mathcal{J}}$ and therefore $\mathcal{J} \models \phi$.

- Let ϕ be of the form $Trans(R)$. Since $\mathcal{I}_i \models \phi$, $R^{\mathcal{I}_i}$ is transitive. Since $R^{\mathcal{I}_i} = R^{\mathcal{J}}$, $R^{\mathcal{J}}$ is transitive and therefore $\mathcal{J} \models \phi$.

Therefore, $\mathcal{J} \models \mathcal{O}$ iff $\mathcal{O}$ is safe.
Q.E.D

Lemma 6 *Let G be a partitioning graph for $\mathcal{O}$ of the form $G = (\{v_1, v_2\}, \{(v_1, v_1), (v_2, v_2)\}, \mathcal{L}, \mathcal{V})$. Let $\mathcal{O}$ be consistent and safe, A a concept with $g(A) = T$ and $\mathcal{V}(A) = v_i$ for some $i \in \{1, 2\}$ and Γ the module for A in $\mathcal{O}$. Then,*

1. $\Gamma \models (A \sqsubseteq B) \Leftrightarrow \mathcal{O} \models (A \sqsubseteq B)$ *for every concept B with $g(B) = T$*
2. $\Gamma \models (B \sqsubseteq A) \Leftrightarrow \mathcal{O} \models (B \sqsubseteq A)$ *for every B with $g(B) = T$*
3. $\Gamma \models A(a) \Leftrightarrow \mathcal{O} \models A(a)$ *for every $a \in V_I$*
4. *A is satisfiable (unsatisfiable) in Γ iff it is satisfiable (unsatisfiable) in $\mathcal{O}$*

Let R be a role $\mathcal{V}(R) = (v_i, v_i)$ for some $i \in \{1, 2\}$ and Σ the module for R in $\mathcal{O}$. Then,

1. $\Sigma \models (R \sqsubseteq S) \Leftrightarrow \mathcal{O} \models (R \sqsubseteq S)$ *for every role S*
2. $\Sigma \models (S \sqsubseteq R) \Leftrightarrow \mathcal{O} \models (S \sqsubseteq R)$ *for every role S*
3. $\Sigma \models R(a, b) \Leftrightarrow \mathcal{O} \models R(a, b)$ *for every pair of individuals a, b in V_I*
4. *R is transitive (respectively functional) in $\mathcal{O} \Leftrightarrow R$ is transitive (respectively functional) in Σ*
5. $\mathcal{O} \models Domain(R, A) \Leftrightarrow \Sigma \models Domain(R, A)$*, with A a concept with $g(A) = T$*
6. $\mathcal{O} \models Range(R, A) \Leftrightarrow \Sigma \models Range(R, A)$*, with A a concept with $g(A) = T$*

Let a be an individual in V_I with $\mathcal{V}(a) = v_i$ for some $i \in \{1, 2\}$ and Δ the module for a in $\mathcal{O}$. Then,

1. $\Delta \models A(a) \Leftrightarrow \mathcal{O} \models A(a)$ *for every concept A with $g(A) = T$*

2. $\Delta \models P(a,b) \Leftrightarrow \mathcal{O} \models P(a,b)$ *for every role* P *and* $b \in V_I$

3. $\Delta \models P(b,a) \Leftrightarrow \mathcal{O} \models P(b,a)$ *for every role* P *and* $b \in V_I$

Proof

We will use the following facts (denoted by $\diamondsuit$):

1. By Theorem 3, $\mathcal{O}$ cannot entail axioms of the following form:
    - $A \sqsubseteq B$, for A, B concepts with $\mathcal{V}(A) \neq \mathcal{V}(B)$ and $g(A) = g(B) = T$
    - $A(a)$ with $g(A) = T$ and $\mathcal{V}(a) \neq \mathcal{V}(C)$.
    - $R \sqsubseteq S$ with R, S roles and $\mathcal{V}(R) \neq \mathcal{V}(S)$.
    - $R(a,b)$ with $\mathcal{V}(R) \neq (\mathcal{V}(a), \mathcal{V}(b))$
2. Let ϕ be any $\mathcal{SHOIN}$ axiom in the vocabulary of $\mathcal{O}$. Let $\Upsilon \subseteq \mathcal{O}$ be any subset of $\mathcal{O}$. Then, if $\Upsilon \models \phi$ we have, by monotonicity of $\mathcal{SHOIN}$, that $\mathcal{O} \models \phi$

Using these facts, the proof reduces to proving the following claim (denoted by $\spadesuit$):
CLAIM ($\spadesuit$):

- $\mathcal{O} \models (A \sqsubseteq B) \Rightarrow \Gamma \models (A \sqsubseteq B)$, with $\mathcal{V}(A) = \mathcal{V}(B)$ and $g(A) = g(B) = T$
- $\mathcal{O} \models (B \sqsubseteq A) \Rightarrow \Gamma \models (A \sqsubseteq B)$, with $\mathcal{V}(A) = \mathcal{V}(B)$ and $g(A) = g(B) = T$
- $\mathcal{O} \models A(a) \Rightarrow \Gamma \models A(a)$ for every $a \in V_I$ with $\mathcal{V}(A) = \mathcal{V}(a)$ and $g(A) = T$
- If A is satisfiable (unsatisfiable) in $\mathcal{O} \Rightarrow A$ is satisfiable (unsatisfiable) in Γ
- $\mathcal{O} \models (R \sqsubseteq S) \Rightarrow \Sigma \models (R \sqsubseteq S)$, with $\mathcal{V}(R) = \mathcal{V}(S)$
- $\mathcal{O} \models (S \sqsubseteq R) \Rightarrow \Sigma \models (S \sqsubseteq R)$, with $\mathcal{V}(R) = \mathcal{V}(S)$
- $\mathcal{O} \models R(a,b) \Rightarrow \Sigma \models R(a,b)$ for $\mathcal{V}(a) = \mathcal{V}(b) = v_i$
- R is transitive (respectively functional) in $\mathcal{O} \Rightarrow$, R is transitive (respectively functional) in Σ
- $\mathcal{O} \models \mathit{Domain}(R,A) \Rightarrow \Sigma \models \mathit{Domain}(R,A)$, with $\mathcal{V}(A) = v_i$
- $\mathcal{O} \models \mathit{Range}(R,A) \Rightarrow \Sigma \models \mathit{Range}(R,A)$, with $\mathcal{V}(A) = v_i$
- $\mathcal{O} \models A(a) \Rightarrow \Delta \models A(a)$ for every concept A with $\mathcal{V}(A) = \mathcal{V}(a)$ and $g(A) = T$
- $\mathcal{O} \models P(a,b) \Rightarrow \Delta \models P(a,b)$ for $\mathcal{V}(P) = (v_i, v_i)$ and $\mathcal{V}(b) = \mathcal{V}(a)$

- $\mathcal{O} \models P(b, a) \Rightarrow \Delta \models P(b, a)$ for $\mathcal{V}(P) = (v_i, v_i)$ and $\mathcal{V}(b) = \mathcal{V}(a)$

Moreover, the following statements hold:

- Γ, Σ and Δ coincide with $\mathcal{L}(v_i)$, by definition of module.
- If $\mathcal{I}_1$ $\mathcal{I}_2$ are models of $\mathcal{L}(v_1)$ and $\mathcal{L}(v_2)$ respectively, then $\mathcal{J}$ defined as in Lemma 8 is a model of $\mathcal{O}$

We will use the above statements to prove $\spadesuit$.

Suppose that $\mathcal{O} \models (A \sqsubseteq B)$ and Γ does not entail $A \sqsubseteq B$. There is a model $\mathcal{I}_i$ of Γ that does not satisfy the axiom. Let $\mathcal{I}_{\bar{i}}$ be a model of $\mathcal{L}(v_{\bar{i}})$ s.t. $\mathcal{W}_{I_i} \cap \mathcal{W}_{I_{\bar{i}}} = \emptyset$, then $\mathcal{J}$ defined as in Lemma 8 is a model of $\mathcal{O}$. Since $\mathcal{V}(A) = \mathcal{V}(B)$, then by Lemma 8, $A^{\mathcal{J}} = A^{\mathcal{I}_i}$ and $B^{\mathcal{J}} = B^{\mathcal{I}_i}$. Hence, since $\mathcal{I}_i$ does not satisfy $A \sqsubseteq B$, nor does $\mathcal{J}$. Since $\mathcal{J}$ is a model of $\mathcal{O}$ we find a contradiction. Analogous arguments apply to the case $B \sqsubseteq A$.

Suppose that $\mathcal{O} \models A(a)$ and Γ does not entail $A(a)$. There is a model $\mathcal{I}_i$ of Γ s.t. $a^{\mathcal{I}_i} \notin A^{\mathcal{I}_i}$. Again, given $\mathcal{I}_{\bar{i}}$ a model of $\mathcal{L}(v_{\bar{i}})$ s.t. $\mathcal{W}_{I_i} \cap \mathcal{W}_{I_{\bar{i}}} = \emptyset$, we have that $\mathcal{J}$ defined as given in Lemma 8 is a model of $\mathcal{O}$. Since $\mathcal{V}(A) = \mathcal{V}(a)$, then by Lemma 8, $A^{\mathcal{J}} = A^{\mathcal{I}_i}$ and $a^{\mathcal{J}} = a^{\mathcal{I}_i}$. Since $a^{\mathcal{I}_i} \notin A^{\mathcal{I}_i}$, then $a^{\mathcal{J}} \notin A^{\mathcal{J}}$, which yields a contradiction.

Suppose that A is satisfiable in $\mathcal{O}$ and unsatisfiable in Γ. This cannot happen by monotonicity of $\mathcal{SHOIN}$.

Suppose that A is unsatisfiable in $\mathcal{O}$ and satisfiable in Γ. If A is satisfiable in Γ, then there is a model $\mathcal{I}_i$ of Γ s.t. $A^{\mathcal{I}_i} \neq \emptyset$. Again, we take a model $\mathcal{I}_{\bar{i}}$ of $\mathcal{L}(v_{\bar{i}})$ and construct $\mathcal{J}$ as before and $A^{\mathcal{J}} = A^{\mathcal{I}_i}$. Hence, $A^{\mathcal{J}} \neq \emptyset$, which implies that A is satisfiable in $\mathcal{O}$, which yields a contradiction.

Suppose that $\mathcal{O} \models (R \sqsubseteq S)$ and Σ does not entail $R \sqsubseteq S$. There is a model $\mathcal{I}_i$ of Σ that does not satisfy the axiom. Let $\mathcal{I}_{\bar{i}}$ be a model of $\mathcal{L}(v_{\bar{i}})$ s.t. $\mathcal{W}_{I_i} \cap \mathcal{W}_{I_{\bar{i}}} = \emptyset$, then $\mathcal{J}$ defined as in Lemma 8 is a model of $\mathcal{O}$. Since $\mathcal{V}(R) = \mathcal{V}(S)$, then by Lemma 8, $R^{\mathcal{J}} = R^{\mathcal{I}_i}$ and $S^{\mathcal{J}} = S^{\mathcal{I}_i}$. Hence, since $\mathcal{I}_i$ does not satisfy $R \sqsubseteq S$, nor does $\mathcal{J}$. Since $\mathcal{J}$ is a model of $\mathcal{O}$ we find a contradiction. Analogous arguments apply to the case $S \sqsubseteq R$.

Suppose that $\mathcal{O} \models R(a, b)$ and Σ does not. There is a model $\mathcal{I}_i$ of Σ that does not satisfy the axiom. Let $\mathcal{I}_{\bar{i}}$ be a model of $\mathcal{L}(v_{\bar{i}})$ s.t. $\mathcal{W}_{I_i} \cap \mathcal{W}_{I_{\bar{i}}} = \emptyset$, then $\mathcal{J}$ defined as in Lemma 9 is a model of $\mathcal{O}$. Since $\mathcal{V}(R) = (v_i, v_i), \mathcal{V}(a) = \mathcal{V}(b) = v_i$, then by Lemma 8, $R^{\mathcal{J}} = R^{\mathcal{I}_i}$, $a^{\mathcal{J}} = a^{\mathcal{I}_i}$ and $b^{\mathcal{J}} = b^{\mathcal{I}_i}$. Hence, since $\mathcal{I}_i$ does not satisfy $R \sqsubseteq S$, nor does $\mathcal{J}$. Since $\mathcal{J}$ is a model of $\mathcal{O}$ we find a contradiction.

Let R be functional in $\mathcal{O}$. It is easy to see that R is functional in $\mathcal{O}$ (respectively in Σ) iff the concept $\geq 2R$ is unsatisfiable in $\mathcal{O}$ (respectively in Σ). However, since $g(\geq 2R) = T$, we have already proved that $\geq 2R$ is unsatisfiable in $\mathcal{O}$ iff it is unsatisfiable in Σ. The same argument applies to the cases of Domain and Range. It is easy to see that $\mathcal{O} \models \mathit{Domain}(R, A)$ iff $\mathcal{O} \models (\exists R.\top \sqsubseteq A)$. Since $g(\exists R.\top) = g(A) = T$, we have already proved that $\mathcal{O} \models (\exists R.\top \sqsubseteq A) \Leftrightarrow \Sigma \models (\exists R.\top \sqsubseteq A)$. Similarly, for the case of Range, $\mathcal{O} \models \mathit{Range}(R, A)$ iff $\exists R.\neg A$ unsatisfiable in $\mathcal{O}$.

Again, since $g(\exists R.\neg A) = T$, we have proved that $\exists R.\neg A$ is unsatisfiable in $\mathcal{O}$ iff it is unsatisfiable in Σ.

Finally, suppose that R is transitive in $\mathcal{O}$, but not in Σ, then there is a model $\mathcal{I}_i$ of Σ for which there are individuals $a, b, c \in \mathcal{W}_i$ that verify the following:

$$R(a, b) \text{ and } R(b, c) \text{ in } R^{\mathcal{I}_i}, \text{ but } (a, c) \notin R^{\mathcal{I}_i}$$

Let $\mathcal{I}_{\bar{i}}$ be a model of $\mathcal{L}(v_{\bar{i}})$ s.t. $\mathcal{W}_{I_i} \cap \mathcal{W}_{I_{\bar{i}}} = \emptyset$, then $\mathcal{J}$ defined as in Lemma 8 is a model of $\mathcal{O}$, $R^{\mathcal{J}} = R^{\mathcal{I}_i}$ and $\mathcal{W}_J = \mathcal{W}_{I_i} \cup \mathcal{W}_{I_{\bar{i}}}$. Hence $a, b, c \in \mathcal{W}_J$ and $R(a, b)$, $R(b, c)$ in $R^{\mathcal{J}}$, but $(a, c) \notin R^{\mathcal{J}}$. Therefore R is not transitive in $\mathcal{O}$, which is a contradiction.

Finally, the proofs for the last three bullets of ♠, concerning the individual a, straightforwardly reduce to cases for concepts and roles that we have already proved.

Q.E.D

Lemma 7 *Let G be a partitioning graph for a consistent ontology $\mathcal{O}$ of the form:*

$$G = (\{v_1, v_2\}, \{(v_1, v_1), (v_1, v_2), (v_2, v_2)\}, \mathcal{L}, \mathcal{V})$$

such that there is no nominal concept N with $\mathcal{V}(N) = v_2$ and there is no assertion in $\mathcal{L}(v_1)$ of the form $R(a, b)$ with $\mathcal{V}(R) = (v_1, v_2)$.

Let $\mathcal{I} = (\mathcal{W}_I, .^{\mathcal{I}})$ be a model of $\mathcal{L}(v_2)$ and $\mathcal{J} = (\mathcal{W}_J, .^{\mathcal{J}})$ be a model of $\mathcal{L}(v_1) \cup \mathcal{T}_2$ (the TBox of $\mathcal{L}(v_2)$) s.t. $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$.

Then, the interpretation $\mathcal{M} = (\mathcal{W}_M, .^{\mathcal{M}})$ defined as follows:

- $\mathcal{W}_M = \mathcal{W}_I \cup \mathcal{W}_J$
- $A^{\mathcal{M}} = A^{\mathcal{I}} \cup A^{\mathcal{J}}$ *if* $\mathcal{V}(A) = v_2$ *and* $A^{\mathcal{M}} = A^{\mathcal{J}}$ *if* $\mathcal{V}(a) = v_1$
- $a^{\mathcal{M}} = a^{\mathcal{J}}$ *if* $\mathcal{V}(a) = v_1$ *and* $a^{\mathcal{M}} = a^{\mathcal{I}}$ *if* $\mathcal{V}(a) = v_2$
- $R^{\mathcal{M}} = R^{\mathcal{J}}$ *if* $\mathcal{V}(R) = (v_1, v_1)$ *or* $\mathcal{V}(R) = (v_1, v_2)$; $R^{\mathcal{M}} = R^{\mathcal{J}} \cup R^{\mathcal{I}}$ *if* $\mathcal{V}(R) = (v_2, v_2)$

is a model of $\mathcal{O}$ if $\mathcal{O}$ is safe.

Proof

First, we show the following (♣):

1. For every concept C with $\mathcal{V}(C) = v_1$, $C^{\mathcal{M}} = C^{\mathcal{J}}$ if $g(C) = T$ and $C^{\mathcal{M}} = C^{\mathcal{J}} \cup \mathcal{W}_I$ if $g(C) = F$.

2. For every concept C with $\mathcal{V}(C) = v_2$, $C^{\mathcal{M}} = C^{\mathcal{J}} \cup C^{\mathcal{I}}$.

Proof for $\clubsuit$:

Let $\mathcal{V}(C) = v_2$. First, note that if $\mathcal{V}(R) = (v_2, v_2)$, for a role R (atomic or not), $R^{\mathcal{M}} = R^{\mathcal{I}} \cup R^{\mathcal{J}}$. We proceed by induction. At the base of the induction we have:

- If C is an atomic concept, then $g(C) = T$ and, by definition of $\mathcal{M}$, $C^{\mathcal{M}} = C^{\mathcal{J}} \cup C^{\mathcal{I}}$.
- If C is $\top$, then $C^{\mathcal{M}} = \mathcal{W}_I \cup \mathcal{W}_J = C^{\mathcal{J}} \cup C^{\mathcal{I}}$ and $g(C) = F$
- Let C be of the form $\geq nR$. In this case, $g(C) = T$. We have that $R^{\mathcal{M}} = R^{\mathcal{I}} \cup R^{\mathcal{J}}$, $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$, $R^{\mathcal{I}} \subseteq \mathcal{W}_I \times \mathcal{W}_I$ and $R^{\mathcal{J}} \subseteq \mathcal{W}_J \times \mathcal{W}_J$. By the semantics of $(\geq nR)$, it is easy to see that $\clubsuit$ holds.
- Let C be of the form $\leq nR$. In this case, $g(C) = F$. Again, we have that $R^{\mathcal{M}} = R^{\mathcal{I}} \cup R^{\mathcal{J}}$, $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$, $R^{\mathcal{I}} \sqsubseteq \mathcal{W}_I \times \mathcal{W}_I$ and $R^{\mathcal{J}} \sqsubseteq \mathcal{W}_J \times \mathcal{W}_J$. By the semantics of $(\leq nR)^{\mathcal{M}}$ it is easy to see that $\clubsuit$ holds.

Note that C **cannot** be a concept of the form $\{a\}$, since $\mathcal{L}(v_2)$ does not contain nominals.

For the induction step we have:

- Let C be of the form $\neg D$. By induction hypothesis, $D^{\mathcal{M}} = D^{\mathcal{J}} \cup D^{\mathcal{I}}$. By the semantics, $C^{\mathcal{M}} = \mathcal{W}_M - D^{\mathcal{M}}$. Hence, $C^{\mathcal{M}} = (\mathcal{W}_J \cup \mathcal{W}_I) - (D^{\mathcal{J}} \cup D^{\mathcal{I}})$. Since $D^{\mathcal{J}} \subseteq \mathcal{W}_J$, $D^{\mathcal{I}} \subseteq \mathcal{W}_I$, and $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$, we have that $C^{\mathcal{M}} = (\mathcal{W}_I - D^{\mathcal{I}}) \cup (\mathcal{W}_J - D^{\mathcal{J}}) = (\neg D)^{\mathcal{I}} \cup (\neg D)^{\mathcal{J}}$, which verifies the induction hypothesis.
- Let C be of the form $D \sqcap E$. By the semantics, $C^{\mathcal{M}} = D^{\mathcal{M}} \cap E^{\mathcal{M}}$. By induction hypothesis we have, $D^{\mathcal{M}} = D^{\mathcal{I}} \cup D^{\mathcal{J}}$ and $E^{\mathcal{M}} = E^{\mathcal{I}} \cup E^{\mathcal{J}}$. Hence, $C^{\mathcal{M}} = (D^{\mathcal{I}} \cup D^{\mathcal{J}}) \cap (E^{\mathcal{I}} \cup E^{\mathcal{J}})$. Since $D^{\mathcal{J}} \subseteq \mathcal{W}_J$, $E^{\mathcal{J}} \subseteq \mathcal{W}_J$, $D^{\mathcal{I}} \subseteq \mathcal{W}_I$, $E^{\mathcal{I}} \subseteq \mathcal{W}_I$ and $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$, we have that $C^{\mathcal{M}} = (D^{\mathcal{J}} \cap E^{\mathcal{J}}) \cup (D^{\mathcal{I}} \cap E^{\mathcal{I}}) = C^{\mathcal{J}} \cup C^{\mathcal{I}}$
- Let C be of the form $D \sqcup E$. By the semantics, $C^{\mathcal{M}} = D^{\mathcal{M}} \cup E^{\mathcal{M}}$. By induction hypothesis we have, $D^{\mathcal{M}} = D^{\mathcal{I}} \cup D^{\mathcal{J}}$ and $E^{\mathcal{M}} = E^{\mathcal{I}} \cup E^{\mathcal{J}}$. Hence, $C^{\mathcal{M}} = (D^{\mathcal{I}} \cup D^{\mathcal{J}}) \cup (E^{\mathcal{I}} \cup E^{\mathcal{J}})$. Reorganizing disjuncts we have $C^{\mathcal{M}} = (D^{\mathcal{I}} \cup E^{\mathcal{I}}) \cup (D^{\mathcal{J}} \cup E^{\mathcal{J}})$. Hence $C^{\mathcal{M}} = C^{\mathcal{I}} \cup C^{\mathcal{J}}$.
- Let C be of the form $\exists R.D$. We have that $R^{\mathcal{M}} = R^{\mathcal{I}} \cup R^{\mathcal{J}}$ and by induction hypothesis $D^{\mathcal{M}} = D^{\mathcal{I}} \cup D^{\mathcal{J}}$, with $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$, $R^{\mathcal{I}} \subseteq \mathcal{W}_I \times \mathcal{W}_I$ and $R^{\mathcal{J}} \subseteq \mathcal{W}_J \times \mathcal{W}_J$. By the semantics of $(\exists R.D)$ it is easy to see that the induction hypothesis holds.

- Let C be of the form $\forall R.D$. Again, we have $R^{\mathcal{M}} = R^{\mathcal{I}} \cup R^{\mathcal{J}}$ and by induction hypothesis $D^{\mathcal{M}} = D^{\mathcal{I}} \cup D^{\mathcal{J}}$, with $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$, $R^{\mathcal{I}} \subseteq \mathcal{W}_I \times \mathcal{W}_I$ and $R^{\mathcal{J}} \subseteq \mathcal{W}_J \times \mathcal{W}_J$. By the semantics of $\forall R.D$, it is easy to verify that the induction hypothesis holds.

Let $\mathcal{V}(C) = v_1$. Note that, if R is a role (atomic or not) s.t. $\mathcal{V}(R) = (v_1, v_1)$, then $R^{\mathcal{M}} = R^{\mathcal{J}}$. Also note that, by construction of $G(\mathcal{O})$ the inverse of a role S s.t. $\mathcal{V}(S) = (v_1, v_2)$ does not occur in $\mathcal{O}$.

We proceed by induction. At the base of the induction we have:

- If C is an atomic concept, then by definition $C^{\mathcal{M}} = C^{\mathcal{J}}$ and $g(C) = T$.
- If C is a concept of the form $\{a\}$, then by definition $a^{\mathcal{M}} = a^{\mathcal{J}}$ and $g(C) = T$
- If C is $\top$, then $C^{\mathcal{M}} = \mathcal{W}_I \cup \mathcal{W}_J = C^{\mathcal{J}} \cup \mathcal{W}_I$ and $g(C) = F$
- C of the form $\geq nR$. In this case $g(C) = T$. Both if $\mathcal{V}(R) = (v_1, v_1)$ or $\mathcal{V}(R) = (v_1, v_2)$, $R^{\mathcal{M}} = R^{\mathcal{J}}$. Given that $\top^{\mathcal{M}} = \mathcal{W}_I \cup \mathcal{W}_J$, $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$ and $R^{\mathcal{J}} \subseteq \mathcal{W}_J \times \mathcal{W}_J$, it is easy to see that $C^{\mathcal{M}} = C^{\mathcal{J}}$
- C of the form $\leq nR$. In this case, $g(C) = F$. Both if $\mathcal{V}(R) = (v_1, v_1)$ or $\mathcal{V}(R) = (v_1, v_2)$, $R^{\mathcal{M}} = R^{\mathcal{J}}$. Given that $\top^{\mathcal{M}} = \mathcal{W}_I \cup \mathcal{W}_J$, $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$ and $R^{\mathcal{J}} \subseteq \mathcal{W}_J \times \mathcal{W}_J$ it is easy to see that $(\leq nR)^{\mathcal{M}} = \mathcal{W}_I \cup (\leq nR)^{\mathcal{J}}$.

For the induction step we have:

- Let C be of the form $\neg D$. Two possibilities:
  - $g(D) = T$, in which case $g(C) = F$. If $g(D) = T$, by induction hypothesis $D^{\mathcal{M}} = D^{\mathcal{J}}$. By the semantics, $C^{\mathcal{M}} = \mathcal{W}_M - D^{\mathcal{M}}$. Hence, $C^{\mathcal{M}} = (\mathcal{W}_J \cup \mathcal{W}_I) - D^{\mathcal{J}}$. Since $D^{\mathcal{J}} \subseteq \mathcal{W}_J$ and $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$, we have that $C^{\mathcal{M}} = \mathcal{W}_I \cup (\mathcal{W}_J - D^{\mathcal{J}}) = \mathcal{W}_I \cup (\neg D)^{\mathcal{J}} = \mathcal{W}_I \cup C^{\mathcal{J}}$
  - $g(D) = F$, in which case $g(C) = T$. If $g(D) = F$, by induction hypothesis $D^{\mathcal{M}} = D^{\mathcal{J}} \cup \mathcal{W}_I$. By the semantics, $C^{\mathcal{M}} = \mathcal{W}_M - D^{\mathcal{M}}$. Hence, $C^{\mathcal{M}} = (\mathcal{W}_J \cup \mathcal{W}_I) - (\mathcal{W}_I \cup D^{\mathcal{J}})$. Since $D^{\mathcal{J}} \subseteq \mathcal{W}_J$ and $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$, we have that $C^{\mathcal{M}} = \mathcal{W}_J - D^{\mathcal{J}} = (\neg D)^{\mathcal{J}}$
- Let C be of the form $D \sqcap E$. Then $C^{\mathcal{M}} = D^{\mathcal{M}} \cap E^{\mathcal{M}}$. By induction hypothesis, $D^{\mathcal{M}}$ is either $D^{\mathcal{J}}$, in which case $g(D) = T$, or $D^{\mathcal{J}} \cup \mathcal{W}_I$, in which case $g(D) = F$. Analogously, $E^{\mathcal{M}}$ is either $E^{\mathcal{J}}$ ($g(E) = T$) or $E^{\mathcal{J}} \cup \mathcal{W}_I$ ($g(E) = F$). Since $D^{\mathcal{J}}$ and $E^{\mathcal{J}}$ are disjoint with $\mathcal{W}_I$, we have, for every possible combination, that $D^{\mathcal{M}} \cap E^{\mathcal{M}}$ is either equal to $D^{\mathcal{J}} \cap E^{\mathcal{J}}$ or to $(D^{\mathcal{J}} \cap E^{\mathcal{J}}) \cup \mathcal{W}_I$. Note that the latter case only occurs if $D^{\mathcal{M}} = D^{\mathcal{J}} \cup \mathcal{W}_I$ and also $E^{\mathcal{M}} = E^{\mathcal{J}} \cup \mathcal{W}_I$. Also note that if $g(D) = g(E) = F$ then $g(C) = F$ and otherwise $g(C) = T$.

- Let C be of the form $D \sqcup E$. Then $(D \sqcup E)^{\mathcal{M}} = D^{\mathcal{M}} \cup E^{\mathcal{M}}$. Again, by induction hypothesis, $D^{\mathcal{M}}$ is either $D^{\mathcal{J}}$, in which case $g(D) = T$, or $D^{\mathcal{J}} \cup \mathcal{W}_I$, in which case $g(D) = F$. Again, for every possible combination we have that $D^{\mathcal{M}} \cup E^{\mathcal{M}}$ is either equal to $D^{\mathcal{J}} \cup E^{\mathcal{J}}$ or to $(D^{\mathcal{J}} \cup E^{\mathcal{J}}) \cup \mathcal{W}_I$. Note that the former case only occurs if $D^{\mathcal{M}} = D^{\mathcal{J}}$ and $E^{\mathcal{M}} = E^{\mathcal{J}}$ and also that $g(C) = T \Leftrightarrow g(D) = T$ and $g(E) = T$.

- If C is of the form $\exists R.D$, then $g(C) = T$ and $(\exists R.D)^{\mathcal{M}} = \{x \in \mathcal{W}_M | \exists y \in \mathcal{W}_M \text{ s.t. } (x, y) \in R^{\mathcal{M}} \text{ and } y \in D^{\mathcal{M}}\}$. We have two cases:
  - $\mathcal{V}(R) = (v_1, v_1)$ and $\mathcal{V}(D) = v_1$. We have $R^{\mathcal{M}} = R^{\mathcal{J}}$. Again two possibilities:
    - $g(D) = T$, which implies that $D^{\mathcal{M}} = D^{\mathcal{J}}$. It is immediate to see that $C^{\mathcal{M}} = C^{\mathcal{J}}$.
    - $g(D) = F$, in which case $D^{\mathcal{M}} = D^{\mathcal{J}} \cup \mathcal{W}_I$. Since $R^{\mathcal{M}} = R^{\mathcal{J}}$, there is no element $y \in \mathcal{W}_M$ s.t. $(x, y) \in R^{\mathcal{M}}$ and $y \in \mathcal{W}_I$, since y must be in $\mathcal{W}_J$. Therefore $C^{\mathcal{M}} = C^{\mathcal{J}}$ and the induction hypothesis holds.
  - $\mathcal{V}(R) = (v_1, v_2)$ and $\mathcal{V}(D) = v_2$. In this case, again $R^{\mathcal{M}} = R^{\mathcal{J}}$ and $D^{\mathcal{M}} = D^{\mathcal{J}} \cup D^{\mathcal{I}}$, since $\mathcal{V}(D) = v_2$. We have that $D^{\mathcal{I}} \subseteq \mathcal{W}_I$ and the proof reduces to the case above.

- In case C is of the form $\forall R.D$ similar considerations as in the case of $(\leq nR)$ apply

We now show that if $\mathcal{O}$ is safe, then $\mathcal{M} \models \mathcal{L}(v_1)$. Suppose that $D \sqsubseteq E$ is a GCI in $\mathcal{L}(v_1)$. If the ontology is safe, the GCI can be only of one of the following three forms:

- $g(D) = g(E) = T$. In this case, $D^{\mathcal{M}} = D^{\mathcal{J}}$ and $E^{\mathcal{M}} = E^{\mathcal{J}}$ and hence if $D^{\mathcal{J}} \subseteq E^{\mathcal{J}}$, then $D^{\mathcal{M}} \subseteq E^{\mathcal{M}}$.

- $g(D) = T$ and $g(E) = F$. In this case, $D^{\mathcal{M}} = D^{\mathcal{J}}$ and $E^{\mathcal{M}} = E^{\mathcal{J}} \cup \mathcal{W}_I$. Hence, if $D^{\mathcal{J}} \subseteq E^{\mathcal{J}}$, then $D^{\mathcal{M}} \subseteq E^{\mathcal{M}}$.

- $g(D) = g(E) = F$. In this case, $D^{\mathcal{M}} = D^{\mathcal{J}} \cup \mathcal{W}_I$ and $E^{\mathcal{M}} = E^{\mathcal{J}} \cup \mathcal{W}_I$. Hence, if $D^{\mathcal{J}} \subseteq E^{\mathcal{J}}$, then $D^{\mathcal{M}} \subseteq E^{\mathcal{M}}$.

Using ♣ it is immediate to verify that the following holds:

- $\mathcal{J} \models R \sqsubseteq S \Rightarrow \mathcal{M} \models R \sqsubseteq S$ for any role inclusion axiom $R \sqsubseteq S$ in $\mathcal{L}(v_1)$ with $\mathcal{V}(R) = \mathcal{V}(S) = (v_1, v_1)$ or $\mathcal{V}(R) = \mathcal{V}(S) = (v_1, v_2)$

- $\mathcal{J} \models D(d) \Rightarrow \mathcal{M} \models D(d)$ for any concept assertion $D(d)$ in $\mathcal{L}(v_1)$

- $\mathcal{J} \models R(a,b) \Rightarrow \mathcal{M} \models R(a,b)$ for any role assertion $R(a,b)$ in $\mathcal{L}(v_1)$ with $\mathcal{V}(R) = (v_1, v_1)$ and $\mathcal{V}(a) = \mathcal{V}(b) = v_1$

Therefore, $\mathcal{M}$ satisfies all the axioms in $\mathcal{L}(v_1)$ if $\mathcal{O}$ is safe.
Using ♣ we now show that $\mathcal{M}$ satisfies all the axioms in $\mathcal{L}(v_2)$.

- Let $D \sqsubseteq E$ in $\mathcal{L}(v_2)$. We have that $\mathcal{I} \models D \sqsubseteq E$ and $\mathcal{J} \models D \sqsubseteq E$, since both $\mathcal{I}, \mathcal{J}$ satisfy all the axioms in $\mathcal{L}(v_2)$. Hence, we have that $D^{\mathcal{I}} \subseteq E^{\mathcal{I}}; D^{\mathcal{J}} \subseteq E^{\mathcal{J}}$. Therefore, $\mathcal{M} \models D \sqsubseteq E$, since $D^{\mathcal{M}} = D^{\mathcal{I}} \cup D^{\mathcal{J}}$ and $E^{\mathcal{M}} = E^{\mathcal{I}} \cup E^{\mathcal{J}}$.

- For any role inclusion axiom $R \sqsubseteq S$ in $\mathcal{L}(v_2)$ with $\mathcal{V}(R) = \mathcal{V}(S) = (v_2, v_2)$, we have that $\mathcal{I} \models R \sqsubseteq S$ and $\mathcal{J} \models R \sqsubseteq S$; hence, $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}; R^{\mathcal{J}} \subseteq S^{\mathcal{J}}$. Therefore, $\mathcal{M} \models R \sqsubseteq S$, since $R^{\mathcal{M}} = R^{\mathcal{I}} \cup R^{\mathcal{J}}$ and $S^{\mathcal{M}} = S^{\mathcal{I}} \cup S^{\mathcal{J}}$.

- For any concept assertion $D(d)$ in $\mathcal{L}(v_2)$, we have that $\mathcal{I} \models D(d)$ and hence $d^{\mathcal{I}} \in D^{\mathcal{I}}$. On the other hand, $D^{\mathcal{M}} = D^{\mathcal{J}} \cup D^{\mathcal{I}}$ and $d^{\mathcal{M}} = d^{\mathcal{I}}$. Hence $d^{\mathcal{M}} \in D^{\mathcal{M}}$ and hence $\mathcal{M} \models D(d)$.

- For any role assertion $R(a,b) \in \mathcal{L}(v_2)$, we have that $\mathcal{V}(R) = (v_2, v_2)$. Since $\mathcal{I} \models R(a,b)$, $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$. Now, since $a^{\mathcal{M}} = a^{\mathcal{I}}$, $b^{\mathcal{M}} = b^{\mathcal{I}}$ and $R^{\mathcal{M}} = R^{\mathcal{I}} \cup R^{\mathcal{J}}$, we have that $(a^{\mathcal{M}}, b^{\mathcal{M}}) \in R^{\mathcal{M}}$ and therefore $\mathcal{M} \models R(a,b)$.

This implies that $\mathcal{M}$ satisfies every axiom in $\mathcal{L}(v_2)$ and hence $\mathcal{M}$ is a model of $\mathcal{O}$.

Q.E.D

Lemma 8 *Let G be a partitioning graph for a consistent ontology $\mathcal{O}$ of the form:*

$$G = (\{v_1, v_2\}, \{(v_1, v_1), (v_1, v_2), (v_2, v_2)\}, \mathcal{L}, \mathcal{V})$$

such that there is no nominal concept N with $\mathcal{V}(N) = v_2$ and there is no assertion in $\mathcal{L}(v_1)$ of the form $R(a,b)$ with $\mathcal{V}(R) = (v_1, v_2)$.

Let $\mathcal{O}$ be safe, A a concept with $g(A) = T$ and $\mathcal{V}(A) = v_2$ and Γ the module for A in $\mathcal{O}$. Then,

1. $\Gamma \models (A \sqsubseteq B) \Leftrightarrow \mathcal{O} \models (A \sqsubseteq B)$ *for every concept B with $g(B) = T$*
2. $\Gamma \models (B \sqsubseteq A) \Leftrightarrow \mathcal{O} \models (B \sqsubseteq A)$ *for every B with $g(B) = T$*
3. $\Gamma \models A(a) \Leftrightarrow \mathcal{O} \models A(a)$ *for every $a \in V_I$*
4. *A is satisfiable (unsatisfiable) in Γ iff it is satisfiable (unsatisfiable) in $\mathcal{O}$*

Let R be a role $\mathcal{V}(R) = (v_2, v_2)$ and Σ the module for R in $\mathcal{O}$. Then,

1. $\Sigma \models (R \sqsubseteq S) \Leftrightarrow \mathcal{O} \models (R \sqsubseteq S)$ *for every role S*

2. $\Sigma \models (S \sqsubseteq R) \Leftrightarrow \mathcal{O} \models (S \sqsubseteq R)$ *for every role* S

3. $\Sigma \models R(a,b) \Leftrightarrow \mathcal{O} \models R(a,b)$ *for every pair of individuals* a, b *in* V_I

4. R *is transitive (respectively functional) in* $\mathcal{O} \Leftrightarrow$, R *is transitive (respectively functional) in* Σ

5. $\mathcal{O} \models Domain(R, A) \Leftrightarrow \Sigma \models Domain(R, A)$, *with* A *a concept with* $g(A) = T$

6. $\mathcal{O} \models Range(R, A) \Leftrightarrow \Sigma \models Range(R, A)$, *with* A *a concept with* $g(A) = T$

Let a *be an individual in* V_I *with* $\mathcal{V}(a) = v_2$ *and* Δ *the module for* a *in* $\mathcal{O}$. *Then,*

1. $\Delta \models A(a) \Leftrightarrow \mathcal{O} \models A(a)$ *for every concept* A *with* $g(A) = T$

2. $\Delta \models P(a,b) \Leftrightarrow \mathcal{O} \models P(a,b)$ *for every role* P *and* $b \in V_I$

3. $\Delta \models P(b,a) \Leftrightarrow \mathcal{O} \models P(b,a)$ *for every role* P *and* $b \in V_I$

Proof

Let $\mathcal{T}_2$ and $\mathcal{A}_2$ be respectively the TBox and the ABox of $\mathcal{L}(v_2)$. Since $\mathcal{L}(v_2)$ does not contain nominals, then:

$$\mathcal{L}(v_2) \models C \sqsubseteq D \Leftrightarrow \mathcal{T}_2 \models C \sqsubseteq D$$

The facts $\diamondsuit$ in Lemma 8 also hold in this case. Hence, the proof of the Lemma reduces to claim $\spadesuit$ in Lemma 8 (with $v_i = v_2$), that we now must prove for this case.

For such a purpose, we will use the following staments:

- Γ, Σ and Δ coincide with $\mathcal{L}(v_2)$, by definition of module.
- Let $\mathcal{I} = (\mathcal{W}_I, .^{\mathcal{I}})$ be a model of $\mathcal{L}(v_2)$ and $\mathcal{J} = (\mathcal{W}_J, .^{\mathcal{J}})$ be a model of $\mathcal{L}(v_1) \cup \mathcal{T}_2$ (the TBox $\mathcal{T}_2$ of $\mathcal{L}(v_2)$) s.t. $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$. Then, the interpretation $\mathcal{M} = (\mathcal{W}_M, .^{\mathcal{M}})$ defined as given in Lemma 9 is a model of $\mathcal{O}$

Suppose $\mathcal{O} \models A \sqsubseteq B$ and Γ does not. Then there is a model $\mathcal{I} = (\mathcal{W}_I, .^{\mathcal{I}})$ of $\mathcal{L}(v_2)$ that does not satisfy the axiom. We take a model $\mathcal{J}$ of $\mathcal{L}(v_1) \cup \mathcal{T}_2$. This is always possible, since $\mathcal{T}_2$ does not have nominals. Then, we construct the model $\mathcal{M}$ of $\mathcal{O}$ as given in Lemma 9. Since $\mathcal{I}$ does not satisfy $A \sqsubseteq B$, there is an element $x \in A^{\mathcal{I}}$ s.t. $x \notin B^{\mathcal{I}}$. We also have that $\mathcal{M}$ satisfies $A \sqsubseteq B$ iff $A^{\mathcal{M}} \subseteq B^{\mathcal{M}}$, i.e. iff (using again Lemma 9) $A^{\mathcal{I}} \cup A^{\mathcal{J}} \subseteq B^{\mathcal{I}} \cup B^{\mathcal{J}}$. Let $x \in A^{\mathcal{I}}$ s.t. $x \notin B^{\mathcal{I}}$. Since $x \in A^{\mathcal{I}}$, we have that $x \in \mathcal{W}_I$. We also have that $B^{\mathcal{J}} \subseteq \mathcal{W}_J$ and $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$. Hence $x \notin B^{\mathcal{J}}$ and therefore $\mathcal{M}$ does not satisfy the axiom, which yields a contradiction. The same argument applies to the cases $B \sqsubseteq A$, $R \sqsubseteq S$ and $S \sqsubseteq R$.

Suppose $\mathcal{O} \models A(a)$ and Γ does not. Then there is a model $\mathcal{I} = (\mathcal{W}_I, .^{\mathcal{I}})$ of $\mathcal{L}(v_2) = \Gamma$ s.t. $a^{\mathcal{I}} \notin A^{\mathcal{I}}$. Again, we select $\mathcal{J}$ and build $\mathcal{M}$, which is a model of $\mathcal{O}$, as in Lemma 9. $\mathcal{M}$ satisfies $A(a)$ iff $a^{\mathcal{M}} \in A^{\mathcal{M}}$. We have that $a^{\mathcal{M}} = a^{\mathcal{I}}$ and $A^{\mathcal{M}} = A^{\mathcal{I}} \cup A^{\mathcal{J}}$. Since $a^{\mathcal{I}} \in \mathcal{W}_I$, $a^{\mathcal{I}} \notin A^{\mathcal{I}}$ and $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$, we have that $a^{\mathcal{M}} \notin A^{\mathcal{M}}$ and we obtain a contradiction.

The case of role functionality, domain and range reduce to GCIs $C \sqsubseteq D$ of the form $g(C) = g(D) = T$. This case has been proved already.

Suppose $\mathcal{O} \models R(a, b)$, with $\mathcal{V}(R) = (v_2, v_2)$ and $\mathcal{V}(a) = \mathcal{V}(b) = v_2$, but Σ does not. Then, there is a model $\mathcal{I}$ of $\mathcal{L}(v_2) = \Sigma$ s.t. $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$. Again, we select $\mathcal{J}$ and build $\mathcal{M}$, which is a model of $\mathcal{O}$, as in Lemma 9. $\mathcal{M}$ satisfies $R(a, b)$ iff $(a^{\mathcal{M}}, b^{\mathcal{M}}) \in R^{\mathcal{M}}$. We have that $a^{\mathcal{M}} = a^{\mathcal{I}}, b^{\mathcal{M}} = b^{\mathcal{I}}$ and $R^{\mathcal{M}} = R^{\mathcal{I}} \cup R^{\mathcal{J}}$. We have that $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$, by assumption and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{J}}$, because $R^{\mathcal{J}} \subseteq \mathcal{W}_J \times \mathcal{W}_J$ and $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$.

Finally, suppose that R is transitive in $\mathcal{O}$, but not in Σ, then there is a model $\mathcal{I}$ of Σ for which there are individuals $a, b, c \in \mathcal{W}_I$ that verify the following:

$$R(a, b) \text{ and } R(b, c) \text{ in } R^{\mathcal{I}}, \text{ but } (a, c) \notin R^{\mathcal{I}}$$

We take $\mathcal{J}$ and build $\mathcal{M}$ as given in Lemma 9. We have that $R^{\mathcal{M}} = R^{\mathcal{I}} \cup R^{\mathcal{J}}$ and $\mathcal{W}_M = \mathcal{W}_I \cup \mathcal{W}_J$ with $\mathcal{W}_I \cap \mathcal{W}_J = \emptyset$. Therefore $a, b, c \in \mathcal{W}_M$, with $(a, b) \in R^{\mathcal{M}}$ and $(b, c) \in R^{\mathcal{M}}$, but $(a, c) \notin R^{\mathcal{M}}$ (because $(a, c) \notin R^{\mathcal{I}}$ and $(a, c) \notin R^{\mathcal{J}}$. Again, we obtain a contradiction, because $\mathcal{M} \models \mathcal{O}$.

Finally, the proofs for the last three bullets of ♠, concerning the individual a, straightforwardly reduce to cases for concepts and roles that we have already proved.

Q.E.D

Theorem 4 *The module $\Gamma \subseteq O$ for an atomic concept $A \in V_C$ in $\mathcal{O}$ verifies the following:*

1. $\Gamma \models (A \sqsubseteq B) \Leftrightarrow \mathcal{O} \models (A \sqsubseteq B)$ *for every* $B \in V_C$
2. $\Gamma \models (B \sqsubseteq A) \Leftrightarrow \mathcal{O} \models (B \sqsubseteq A)$ *for every* $B \in V_C$
3. $\Gamma \models A(a) \Leftrightarrow \mathcal{O} \models A(a)$ *for every* $a \in V_I$
4. *A is satisfiable (unsatisfiable) in Γ iff it is satisfiable (unsatisfiable) in $\mathcal{O}$*

The module Γ for a role $P \in V_R$ in $\mathcal{O}$ verifies the following:

1. $\Gamma \models (P \sqsubseteq Q) \Leftrightarrow \mathcal{O} \models (P \sqsubseteq Q)$ *for every* $Q \in V_R$
2. $\Gamma \models (Q \sqsubseteq P) \Leftrightarrow \mathcal{O} \models (Q \sqsubseteq P)$ *for every* $Q \in V_R$
3. $\Gamma \models Domain(P, A) \Leftrightarrow \mathcal{O} \models Domain(P, A)$ *for every* $A \in V_C$

4. $\Gamma \models Range(P, A) \Leftrightarrow \mathcal{O} \models Range(P, A)$ *for every* $A \in V_C$
5. $\Gamma \models Transitive(P) \Leftrightarrow \mathcal{O} \models Transitive(P)$
6. $\Gamma \models Functional(P) \Leftrightarrow \mathcal{O} \models Functional(P)$
7. $\Gamma \models P(a, b) \Leftrightarrow \mathcal{O} \models P(a, b)$ *for every* $a, b \in V_I$

The module Γ *for an individual* $a \in V_I$ *in* $\mathcal{O}$ *verifies the following:*

1. $\Gamma \models A(a) \Leftrightarrow \mathcal{O} \models A(a)$ *for every* $A \in V_C$
2. $\Gamma \models P(a, b) \Leftrightarrow \mathcal{O} \models P(a, b)$ *for every* $P \in V_R, b \in V_I$
3. $\Gamma \models P(b, a) \Leftrightarrow \mathcal{O} \models P(b, a)$ *for every* $P \in V_R, b \in V_I$

Proof

Let $G = Partition(\mathcal{O})$, and Γ the module for X in $\mathcal{O}$. Assume $\Gamma = \mathcal{L}(v_1) \cup ... \cup \mathcal{L}(v_k)$ and $G' = Collapse(G, \{v_1, ..., v_k\})$. We have seen that G' can be of either of the following forms:

1. It contains a single node v with $\mathcal{L}(v) = \mathcal{O}$.
2. It contains two nodes v_1, v_2 s.t. $\mathcal{L}(v_2)$ is the module for X in $\mathcal{O}$ and either of the following holds:
   (a) v_1 and v_2 are disconnected ($(v_1, v_2) \notin \mathbf{V}$ and $(v_2, v_1) \notin \mathbf{V}$
   (b) v_1 and v_2 are only connected by an edge (v_1, v_2)

In case 1), the theorem obviously holds. Case 2.1) is a direct consequence of Lemma 8 and case 2.2) of Lemma 10.
Q.E.D

A.4 Complexity

Theorem 5 ***(Complexity)***
The algorithms described in Figures 1 and 7 are worst-case quadratic in the size of the input ontology.

Proof

The definition of safety is structural and it can be easily seen that safety checking can be performed by an algorithm that is quadratic on the size of the input ontology.

The complexity of $Partition(\mathcal{O})$ is justified by the fact that each concept, role or individual X occurring in the input ontology is assigned a new value $\mathcal{V}(X)$ in the graph only once. Similarly an axiom is removed from the label of a node and added to the label of another only once as well. Finally, Each partitioning step involves checks that are linear in the size of the input.

The complexity of $GenerateModule(G(\mathcal{O}), X)$ is linear in the size of the partitioning graph $G(\mathcal{O})$.

Q.E.D